Shorter stabilizer circuits via Bruhat decomposition and quantum circuit transformations

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Abstract— In this paper we improve the layered implementation of arbitrary stabilizer circuits introduced by Aaronson and Gottesman in Phys. Rev. A 70(052328), 2004: to implement a general stabilizer circuit, we reduce their 11-stage computation -H-C-P-C-P-C-H-P-C-P-C- over the gate set consisting of Hadamard, Controlled-NOT, and Phase gates, into a 7-stage computation of the form -C-CZ-P-H-P-CZ-C-. We show arguments in support of using -CZ- stages over the -C- stages: not only the use of -CZ- stages allows a shorter layered expression, but -CZ- stages are simpler and appear to be easier to implement compared to the -C- stages. Based on this decomposition, we develop a two-qubit gate depth-$(14n-4)$ implementation of stabilizer circuits over the gate library $\{H, P, CNOT\}$, executable in the LNN architecture, improving best previously known depth-$25n$ circuit, also executable in the LNN architecture. Our constructions rely on Bruhat decomposition of the symplectic group and on folding arbitrarily long sequences of the form $(\text{-P-C-})^m$ into a 3-stage computation -P-CZ-C-. Our results include the reduction of the 11-stage decomposition -H-C-P-C-P-C-H-P-C-P-C- into a 9-stage decomposition of the form -C-P-C-P-H-C-P-C-P-. This reduction is based on the Bruhat decomposition of the symplectic group. This result also implies a new normal form for stabilizer circuits. We show that a circuit in this normal form is optimal in the number of Hadamard gates used. We also show that the normal form has an asymptotically optimal number of parameters.

I. INTRODUCTION

Stabilizer circuits are of particular interest in quantum information processing (QIP) due to their prominent role in fault tolerance [7], [13], [20], [22], the study of entanglement [4], [20], and in evaluating quantum information processing proposals via randomized benchmarking [16], to name a few.

Stabilizer circuits are composed of the Hadamard gate H, Phase gate P, and the controlled-NOT gate CNOT defined as

$$\text{H} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{P} := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \text{ and CNOT} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where in case of H and P each of these gates is allowed to act on any of a given number n of qubits, and on any pair of qubits in case of the CNOT gate.

The stabilizer circuits over n qubits, such as defined above form a finite group which is known to be equivalent [7], [8] to the group of binary $2n \times 2n$ symplectic matrices, $\text{Sp}(2n, \mathbb{F}_2)$. Knowing this equivalence allows to evaluate the stabilizer group size, through employing the well-known formula to calculate the number of elements in the respective symplectic group,

$$|\text{Sp}(2n, \mathbb{F}_2)| = 2^{n^2} \prod_{j=1}^{n} (2^{2j} - 1) = 2^{2n^2 + O(n)}.$$

In this paper, we rely on the phase polynomial representation of $\{P, CNOT\}$ circuits. Specifically, arbitrary quantum circuits over P and CNOT gates can be described in an alternate form, which we refer to as *phase polynomial description*, and vice versa, each phase polynomial description can be written as a P and CNOT gate circuit. We use this result to induce circuit transformations via rewriting the respective phase polynomials. We adopt the phase polynomial expression result from [2] to this paper as follows:

Theorem 1: Any circuit C on n qubits over $\{P, CNOT\}$ library with k Phase gates can be described by the combination of a phase polynomial $p(x_1, x_2, ..., x_n) = f_1(x_1, x_2, ..., x_n) + f_2(x_1, x_2, ..., x_n) + \cdots + f_k(x_1, x_2, ..., x_n)$ and a linear reversible function $g(x_1, x_2, ..., x_n)$, such that the action of C can be constructed as

$$C|x_1 x_2 ... x_n\rangle = i^{p(x_1, x_2, ..., x_n)} |g(x_1, x_2, ..., x_n)\rangle,$$

where i is the complex number i. Functions f_j corresponding to the jth Phase gate are obtained from the circuit C via devising Boolean linear functions computed by the CNOT gates in the circuit C leading to the position of the respective Phase gate.

In the following we focus on finding a short layered sequence of gates capable of representing an arbitrary stabilizer circuit over n primary inputs. The layers are defined as follows:

- -H- layer contains all unitaries representable by arbitrary circuits composed of the Hadamard gates. Since $\text{H}^2 = Id$, and Hadamard gate is a single-qubit gate, -H- layer has zero or one gates acting on each of the respective qubits. The number of distinct layers -H- on n qubits is thus 2^n. We say -H- has n Boolean degrees of freedom.
- -P- layer is composed of an arbitrary set of Phase gates. Since $\text{P}^4 = Id$, and the Phase gate is also a single-qubit gate, -P- layer has anywhere between zero to three gates on each of the respective qubits. Note that $\text{P}^2 = Z$, and therefore the gate sequence PP may be better implemented as the Pauli-Z gate; $\text{P}^3 = \text{P}^\dagger$, and frequently $\text{P}^\dagger$ is constructible with the same cost as P. This means that the -P- layer is essentially analogous to the -H- layer in the sense that it consists of at most n individual single-qubit gates. The number of different unitaries represented

by -P- layers on n qubits is 2^{2n}. We say -P- has $2n$ Boolean degrees of freedom.

- -C- layer contains all unitaries computable by the CNOT gates. The number of different -C- layers corresponds to the number of affine linear reversible functions, and it is equal to $\prod_{j=0}^{n-1}(2^n - 2^j) = 2^{n^2+O(n)}$ [21]. We say -C- has $n^2 + O(n)$ Boolean degrees of freedom.

- -CZ- layer contains all unitaries computable by the CZ gates, where CZ gate is defined as

$$\mathrm{CZ} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since all CZ gates commute, and due to CZ being self-inverse, *i.e.*, $\mathrm{CZ}^2 = Id$, the number of different unitaries computable by -CZ- layers is $\prod_{j=1}^{n} 2^{n-j} = 2^{\frac{n^2}{2}+O(n)}$. We say -CZ- has $\frac{n^2}{2} + O(n)$ Boolean degrees of freedom.

Observe that the above count of the degrees of freedom suggests that -P- and -H- layers are "simple". Indeed, each requires no more than the linear number of single-qubit gates to be constructed via a circuit. The number of the degrees of freedom in -C- and -CZ- stages is quadratic in n. Other than two-qubit gates often being more expensive than the single-qubit gates [10], [15], the comparison of the degrees of freedom suggests that we will need more of the respective gates to construct each such stage. The -CZ- layer has roughly half the number of the degrees of freedom compared to the -C- layer. We may thus reasonably expect that the -CZ- layer can be easier to obtain.

Unlike the -C- circuits, the problem of optimizing -CZ- circuits does not seem to have been studied in the literature. Part of the reason could be due to the CZ gate complexity of -CZ- circuits being a very inconspicuous problem to study: indeed, worst case optimal circuit has $\frac{(n-1)n}{2}$ CZ gates, and optimal circuits are easy to construct, as they are determined by the presence or lack of CZ gates acting on the individual pairs of qubits. However, we claim that using only CZ gates to construct -CZ- layer is not the best solution, and a better approach would be to also employ the CNOT and P gates. Indeed, both CNOT and CZ gates must have a comparable cost of the implementation, since they are related by the formula $\mathrm{CNOT}(a,b) = \mathrm{H}(b)\mathrm{CZ}(a,b)\mathrm{H}(b)$, and single-qubit gates are "easy" [10], [15]. CZ is furthermore the elementary gate in superconducting circuits QIP [11], and as such, technically, it costs less than the CNOT, and in the trapped ions QIP the costs of the two are comparable [19]. Further discussion of the relation of implementation costs between -C- and -CZ- layers is postponed to Section III.

The different layers can be interleaved to obtain stabilizer circuits not computable by a single layer. A remarkable result of [1] shows that 11 stages over a computation of the form -H-C-P-C-P-C-H-P-C-P-C- suffices to compute an arbitrary stabilizer circuit. The number of Boolean degrees of freedom in the group of stabilizer unitaries, defined as the logarithm base-2 of their total count, is given by the formula $\log_2|\mathrm{Sp}(2n,\mathbb{F}_2)| = 2n^2 + O(n)$. This suggests that the 11-stage circuit by Aaronson and Gottesman [1] is suboptimal,

as it relies on $5n^2 + O(n)$ degrees of freedom, whereas only $2n^2 + O(n)$ are necessary. Indeed, we find (Section VI) a shorter 9-stage decomposition of the form -P-C-P-C-H-C-P-C-P- in which all -C- stages correspond to upper triangular matrices having $n^2/2$ degrees of freedom each, leading to an asymptotically tight parameterization of all stabilizer circuits.

Notation. We denote with $\mathrm{GL}(n,\mathbb{F}_2)$ the group of invertible $n \times n$ matrices, with S_n the full permutation group on n letters, and with $\mathrm{diag}(A,B)$ the (block) diagonal operator that has diagonal elements A and B.

II. (-P-C-)^m CIRCUITS

In this section we show that an arbitrary length n-qubit computation described by the stages -P-C-P-C-...-P-C- folds into an equivalent three-stage computation -P-CZ-C-.

Theorem 2: (-P-C-)^m = -P-CZ-C-.

Proof: Suppose (-P-C-)^m circuit has $k \leq 3nm$ Phase gates. Name those gates $\mathrm{P}_{j=1..k}$, denote Boolean linear functions they apply phases to as $f_{j=1..k}(x_1,x_2,...,x_n)$, and name the reversible linear function computed by (-P-C-)^m (Theorem 1) as $g(x_1,x_2,...,x_n)$. Phase polynomial computed by the original circuit is $f_1(x_1,x_2,...,x_n) + f_2(x_1,x_2,...,x_n) + ... + f_k(x_1,x_2,...,x_n)$. We will next transform phase polynomial to an equivalent one, that will be easier to write as a compact circuit. To accomplish this, observe that $i^{a+b+c+(a\oplus b)+(a\oplus c)+(b\oplus c)+(a\oplus b\oplus c)} = i^4 = 1$, where a, b, and c are arbitrary Boolean linear functions of the primary variables. This equality can be verified by inspection through trying all 8 possible combinations for Boolean values a, b, and c. The equality can be rewritten as

$$i^{a\oplus b\oplus c} = i^{3a+3b+3c+3(a\oplus b)+3(a\oplus c)+3(b\oplus c)}, \tag{1}$$

suggesting how it will be used. The following algorithm takes $n-2$ steps.

Step n. Consolidate terms in the phase polynomial $f_1(x_1,x_2,...,x_n) + f_2(x_1,x_2,...,x_n) + ... + f_k(x_1,x_2,...,x_n)$ by replacing $uf_j(x_1,x_2,...,x_n) + vf_k(x_1,x_2,...,x_n)$ with $(u+v \bmod 4)f_j(x_1,x_2,...,x_n)$ whenever $f_j = f_k$. Once done, look for $f_j = x_1 \oplus x_2 \oplus ... \oplus x_n$, being the maximal length linear function of the primary inputs. If no such function found, move to the next step. If it is found with a non-zero coefficient u, as an additive term $u(x_1 \oplus x_2 \oplus ... \oplus x_n)$, replace it by the equivalent 6-term mixed arithmetic polynomial $(4-u)x_1 + (4-u)x_2 + (4-u)(x_3 \oplus x_4 \oplus ... \oplus x_n) + (4-u)(x_1 \oplus x_2) + (4-u)(x_1 \oplus x_3 \oplus x_4 \oplus ... \oplus x_n) + (4-u)(x_2 \oplus x_3 \oplus ... \oplus x_n)$. This transformation is derived from eq. (1) by assigning $a = x_1$, $b = x_2$, and $c = x_3 \oplus x_4 \oplus ... \oplus x_n$. Consolidate all equal terms. The transformed phase polynomial is equivalent to the original one in the sense of the overall combination of phases it prescribes to compute, however, it is expressed over linear terms with at most $n-1$ variables.

Step s, $s = (n-1)..3$. From the previous step we have phase polynomial of the form $u_1'f_1'(x_1,x_2,...,x_n) + u_2'f_2'(x_1,x_2,...,x_n) + ... + u_{k'}'f_{k'}'(x_1,x_2,...,x_n)$. By construction it is guaranteed that the functions $f_{j=1..k'}'$ EXOR no more than s literals. For each $f_j' = x_{j_1} \oplus x_{j_2} \oplus ... \oplus x_{j_s}$, with the coefficient $u_j' \not\equiv 0 \bmod 4$ replace this term with the sum of

six terms, each having no more than $s-1$ literals by using eq. (1) and setting a, b, and c to carry linear functions over the non-overlapping non-empty subsets of $\{x_{j_1}, x_{j_2}, ..., x_{j_s}\}$ whose union gives the entire set $\{x_{j_1}, x_{j_2}, ..., x_{j_s}\}$. Value $s=3$ marks the last opportunity to break down a term in the phase polynomial expression into a set of terms over smaller numbers of variables. Upon completion of this step, the linear functions participating in the phase polynomial expression contain at most two literals each.

The transformed phase polynomial description of the original circuit now has the following form: phase polynomial $\sum_{j=1}^{n} u_j x_j + \sum_{j=1}^{n} \sum_{k=j+1}^{n} u_{j,k}(x_j \oplus x_k)$, where $u_{\cdot}, u_{\cdot,\cdot} \in \mathbb{Z}_4$, and the linear reversible function $g(x_1, x_2, ..., x_n)$. We next show how to implement such a unitary as a -P-CZ-C- circuit, focusing separately on the phase polynomial and the linear reversible part. We synthesize individual terms in the phase polynomial as follows.

- For $j = 1..n$, the term $u_j x_j$ is obtained as the single-qubit gate circuit $P^{u_j}(x_j)$;
- For $j = 1..n$, $k = j+1..n$, the term $u_{j,k}(x_j \oplus x_k)$ is obtained as follows:
  - if $u_{j,k} \equiv 2 \bmod 4$, by the circuit $P^2(x_j)P^2(x_k) = Z(x_j)Z(x_k)$;
  - if $u_{j,k} \equiv 1$ or $3 \bmod 4$, by the circuit $P^{u_{j,k}}(x_j)P^{u_{j,k}}(x_k)CZ(x_j, x_k)$.

The resulting circuit contains P and CZ gates; it implements phase polynomial $\sum_{j=1}^{n} u_j x_j + \sum_{j=1}^{n} \sum_{k=j+1}^{n} u_{j,k}(x_j \oplus x_k)$ and the identity linear reversible function. Since all P and CZ gates commute, Phase gates can be collected on the left side of the circuit. This results in the ability to express phase polynomial construction as a -P-CZ- circuit. We conclude the entire construction via obtaining the linear reversible function $g(x_1, x_2, ..., x_n)$ as a -C- stage, with the overall computation described as a -P-CZ-C- circuit. ∎

Note that -P-CZ-C- can also be written as -C-P-CZ-, if one first synthesizes the linear reversible function $g(x_1, x_2, ..., x_n) = (g_1(x_1, x_2, ..., x_n), g_2(x_1, x_2, ..., x_n), ..., g_n(x_1, x_2, ..., x_n))$, and then expresses the phase polynomial in terms of the degree-2 terms over the set $\{g_1, g_2, ..., g_n\}$. Other ways to write such a computation include -CZ-P-C- and -C-CZ-P-, that are obtained from the first two by commuting -P- and -CZ- stages.

Corollary 3: -H-C-P-C-P-C-H-P-C-P-C- [1] = -H-C-CZ-P-H-P-CZ-C-.

III. -C- VS -CZ-

We have previously noted that CNOT and CZ gates have a comparable cost as far as their implementation within some QIP proposals is concerned. In this section, we study $\{P, CZ, CNOT\}$ implementations of stages -C- and -CZ-. The goal is to provide further evidence in support of the statement that -CZ- can be thought of as a simpler stage compared to the -C- stage, and going beyond counting the degrees of freedom argument.

Lemma 4: Optimal quantum circuit over $\{CZ\}$ library for a -CZ- stage has at most $\frac{n(n-1)}{2}$ CZ gates.

Indeed, all CZ gates commute, which limits the expressive power of the circuits over CZ gates. However, once we add the non-commuting CNOT gate, and after that the Phase gate, the situation changes. We can now implement -CZ- circuits more efficiently, such as illustrated by the circuit identities shown in Fig. 1. The unitary implemented by the circuitry shown in Fig. 1 requires 7 CZ gates as a $\{CZ\}$ circuit, 6 gates as a $\{CZ, CNOT\}$ circuit, and only 5 two-qubit gates as a $\{P, CZ, CNOT\}$ circuit. This illustrates that the CNOT and P gates are important in constructing efficient -CZ- circuits.

We may consider adding the P and CZ gates to the $\{CNOT\}$ library in hopes of constructing more efficient circuits implementing the -C- stage. However, as the following lemma shows, this does not help.

Lemma 5: Any $\{P, CZ, CNOT\}$ circuit implementing an element of the layer -C- using a non-zero number of P and CZ gates is suboptimal.

Proof: Each P gate applied to a qubit x can be expressed as a phase polynomial $1 \cdot x$ over the identity reversible linear function. Each CZ gate applied to a set of qubits y and z can be expressed as a phase polynomial $y + z + 3(y \oplus z)$ and the identity reversible function. Removing all P and CZ gates from the given circuit thus modifies only the phase polynomial part of its phase polynomial description. Removing all P and CZ gates from the $\{P, CZ, CNOT\}$ circuit guarantees that the phase polynomial of the resulting circuit equals to the identity, such as required in the -C- stage. This results in the construction of a shorter circuit in cases when the original P and CZ gate count was non-zero. ∎

We next show in Table I optimal counts and upper bounds on the number of gates it takes to synthesize the most difficult function from stages -C- and -CZ- for some small n. Observe how the two-qubit gate counts for the -CZ- stage, when constructed as a circuit over $\{P, CZ, CNOT\}$ library, remain lower than those for the -C- stage.

	-CZ-		-C-	
n	$\{CZ\}$	$\{P, CZ, CNOT\}$	$\{CNOT\}$	$\{P, CZ, CNOT\}$
2	1	1	3	3
3	3	3	6	6
4	6	5	9	9
5	10	7	12	12

TABLE I

GATE COUNTS REQUIRED TO IMPLEMENT ARBITRARY -CZ- AND -C- STAGES FOR SOME SMALL n: OPTIMAL -CZ- STAGE GATE COUNTS AS CIRCUITS OVER $\{CZ\}$, UPPER BOUNDS ON THE TWO-QUBIT GATE COUNT FOR -CZ- OVER $\{P, CZ, CNOT\}$, ACHIEVED BASED ON THE APPLICATION OF IDENTITIES FROM FIG. (1) APPLIED TO CIRCUITS WITH CZ GATES, AND OPTIMAL $\{CNOT\}$ AND $\{P, CZ, CNOT\}$ TWO-QUBIT GATE COUNTS FOR STAGE -C-.

In [21] an asymptotically optimal algorithm for $\{CNOT\}$ synthesis of arbitrary -C- stage functions was reported, that leads to the worst case gate complexity of $O\left(\frac{n^2}{\log n}\right)$. It is possible that an asymptotically optimal algorithm for $\{P, CZ, CNOT\}$ circuits implementing arbitrary -CZ- stage
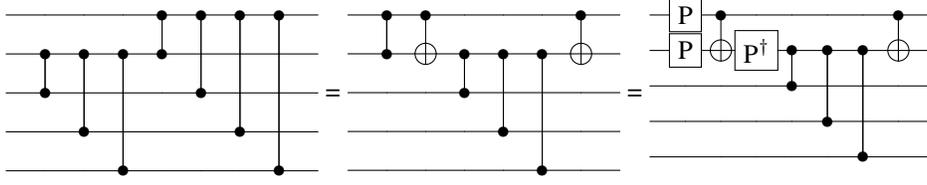
Fig. 1. Circuit identities illustrating rewriting of the -CZ- circuits.

functions can be developed, at which point its complexity has to be $O\left(\frac{n^2}{\log n}\right)$. To determine which of the two results in shorter circuits, one has to develop constants in front of the leading complexity terms.

We point out that gate count is only one of several possible metrics of efficiency. For instance, two-qubit gate depth over Linear Nearest Neighbour (LNN) architecture is also an important metric of efficiency. This metric has been applied in [18] to show an asymptotically optimal upper bound of $5n$ CNOT layers required to obtain an arbitrary -C- stage.

Define -$\widehat{CZ}$- to be -CZ- accompanied by the complete qubit reversal (*i.e.*, the linear reversible mapping $|x_1 x_2...x_n\rangle \mapsto |x_n x_{n-1}...x_1\rangle$). We next show that -$\widehat{CZ}$- can be executed as a two-qubit gate depth-$(2n+2)$ computation over LNN. This result will be used to reduce depth in the implementation of arbitrary stabilizer circuits.

Theorem 6: -$\widehat{CZ}$- can be implemented as a CNOT depth-$(2n+2)$ circuit.

Proof: Consider phase polynomial description of the circuit -$\widehat{CZ}$-. However, rather than describe both parts of the expression, phase polynomial itself and the linear reversible transformation, over the set of primary variables, we will describe phase polynomial over the variables $y_1, y_2, ..., y_n$ defined as follows:

$$y_1 := x_1,$$
$$y_2 := x_1 \oplus x_2,$$
$$...,$$
$$y_n := x_1 \oplus x_2 \oplus ... \oplus x_n.$$

This constitutes the change of basis $\{x_1, x_2, ..., x_n\} \mapsto \{y_1, y_2, ..., y_n\}$. Similarly to how it was done in the proof of Theorem 2, we reduce phase polynomial representation of -$\widehat{CZ}$- to the application of Phase gates to the EXORs of pairs and the individual variables from the set $\{y_1, y_2, ..., y_n\}$,

$$\sum_{j=1}^{n} u_j y_j + \sum_{j=1}^{n} \sum_{k=j+1}^{n} u_{j,k}(y_j \oplus y_k), \qquad (2)$$

and the linear reversible function $g(x_1, x_2, ..., x_n)$: $|x_1 x_2...x_n\rangle \mapsto |x_n x_{n-1}...x_1\rangle$. Observe that $y_j \oplus y_k = x_j \oplus x_{j+1} \oplus ... \oplus x_k$, and thereby this linear function can be encoded by the integer segment $[j,k]$. The primary variable x_j admits the encoding $[j,j]$. We use this notation next. In the following we implement the pair of the phase polynomial expression and the reversal of qubits (a linear reversible function) via a quantum circuit.

Observe that the swapping operation $g(x_1, x_2, ..., x_n)$: $|x_1 x_2...x_n\rangle \mapsto |x_n x_{n-1}...x_1\rangle$ can be implemented as a circuit similar to the one from Theorem 5.1 [18] in depth $2n+2$. The rest of the proof concerns the ability to insert Phase gates in the circuit accomplishing the reversal of qubits such as to allow the implementation of each term in the phase polynomial, eq. (2).

Since our qubit reversal circuit is slightly different from the one used in [18], and we explore its structure more extensively, we describe it next. It consists of $n+1$ alternating stages, S_1 and S_2, where

$$S_1 = \text{CNOT}(x_1; x_2)\text{CNOT}(x_3; x_4)...\text{CNOT}(x_{n-2}; x_{n-1})$$
$$\cdot \ \text{CNOT}(x_3; x_2)\text{CNOT}(x_5; x_4)...\text{CNOT}(x_n; x_{n-1})$$

for odd n, and

$$S_1 = \text{CNOT}(x_1; x_2)\text{CNOT}(x_3; x_4)...\text{CNOT}(x_{n-1}; x_n)$$
$$\cdot \ \text{CNOT}(x_3; x_2)\text{CNOT}(x_5; x_4)...\text{CNOT}(x_{n-1}; x_{n-2})$$

for even n, is a depth-2 circuit composed with the CNOT gates. Similarly,

$$S_2 = \text{CNOT}(x_2; x_1)\text{CNOT}(x_4; x_3)...\text{CNOT}(x_{n-1}; x_{n-2})$$
$$\cdot \ \text{CNOT}(x_2; x_3)\text{CNOT}(x_4; x_5)...\text{CNOT}(x_{n-1}; x_n)$$

for odd n, and

$$S_2 = \text{CNOT}(x_2; x_1)\text{CNOT}(x_4; x_3)...\text{CNOT}(x_n; x_{n-1})$$
$$\cdot \ \text{CNOT}(x_2; x_3)\text{CNOT}(x_4; x_5)...\text{CNOT}(x_{n-2}; x_{n-1})$$

for even n, is also a depth-2 circuit composed with the CNOT gates. We refer to the concatenation of S_1 and S_2 as S. The goal is to show that after $\lceil \frac{n}{2} \rceil$ applications of the circuit S we are able to cycle through all $\frac{n(n+1)}{2}$ linear functions $[j,k]$, $j \leq k$.

The remainder of the proof works slightly differently depending on the parity of n. First, choose odd $n = 2m+1$. Consider two patterns of length $2n-3$,

$$Pj = (n-1, n-3, n-3, ..., 4, 4, 2, 2, 1, 1, 3, 3, ..., n-2, n-2)$$

and

$$Pk = (3, 3, 5, 5, ..., n, n, n-1, n-1, n-3, n-3, ...6, 6, 4, 4, 2).$$

Observe by inspection that the ith linear function computed by the single application of the stage S is given by the formula $[Pj(n-3+i), Pk(i)]$, where $Pj(l)$ and $Pk(l)$ return l^{th} component of the respective pattern. It may further be observed, via direct multiplication by the linear reversible matrix corresponding to the transformation S, that the ith component upon t ($t \leq m$) applications of the circuit S

```
          [6, ]                    [ , ]                    [ ,3]
          [4, ]                    [ , ]                    [ ,3]
          [4, ]                    [6,3]                    [ ,5]
          [2, ]                    [4,3]                    [ ,5]
[1,1] —•——⊕— [2,3] —•——⊕— [4,5] —•——⊕— [6,7] —•——⊕— [7,7]
[2,2] —⊕—⊕—•—•— [1,3] —⊕—⊕—•—•— [2,5] —⊕—⊕—•—•— [4,7] —⊕—⊕—•—•— [6,6]
[3,3] —•———⊕—⊕— [1,5] —•———⊕—⊕— [2,7] —•———⊕—⊕— [4,6] —•———⊕—⊕— [5,5]
[4,4] —⊕—⊕——— [3,5] —⊕—⊕——— [1,7] —⊕—⊕——— [2,6] —⊕—⊕——— [4,4]
[5,5] —•—•—⊕—⊕— [3,7] —•—•—⊕—⊕— [1,6] —•—•—⊕—⊕— [2,4] —•—•—⊕—⊕— [3,3]
[6,6] —⊕—⊕——•— [5,7] —⊕—⊕——•— [3,6] —⊕—⊕——•— [1,4] —⊕—⊕——•— [2,2]
[7,7] ————•——⊕— [5,6] ————•——⊕— [3,4] ————•——⊕— [1,2] ————•——⊕— [1,1]
          [ ,6]                    [5,4]                    [3, ]
          [ ,4]                    [5,2]                    [3, ]
          [ ,4]                    [ , ]                    [5, ]
          [ ,2]                    [ , ]                    [5, ]
           ↓ ↑                      ↓ ↑                      ↓ ↑
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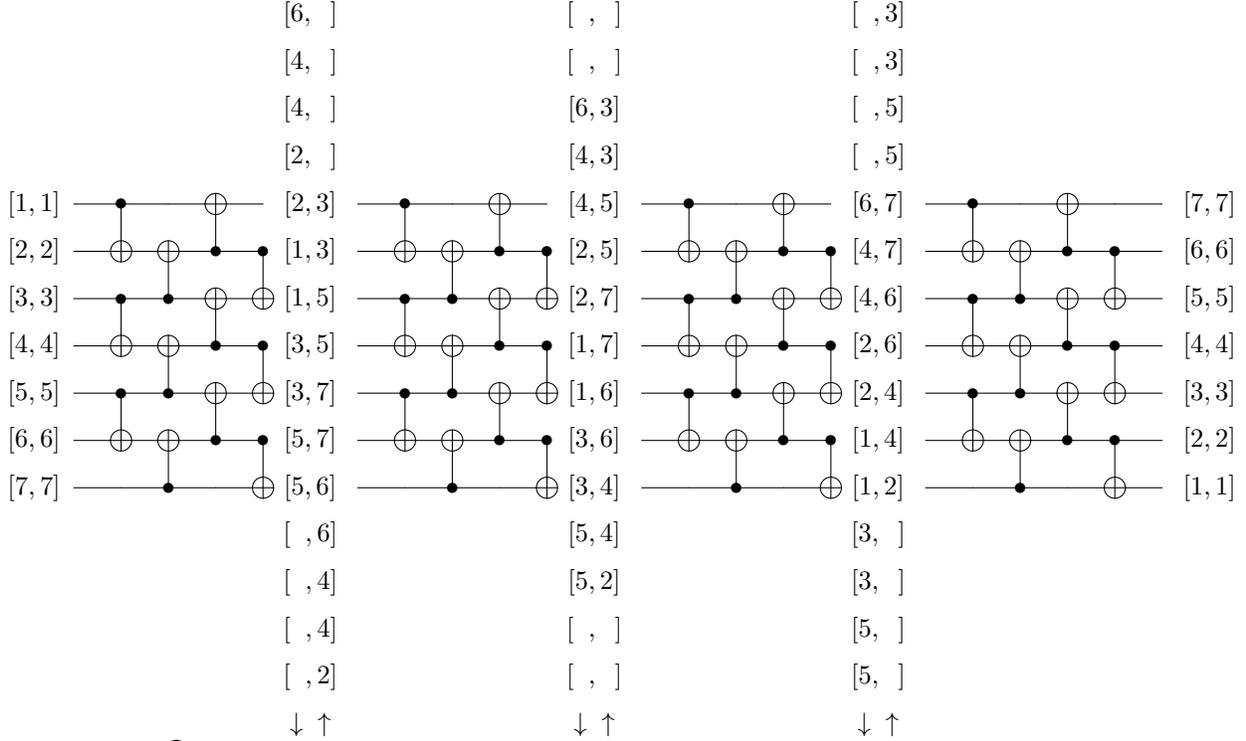
Fig. 2. Constructing -$\widehat{\text{CZ}}$- for $n = 7$. The circuit uses $m + 1 = \frac{n-1}{2} + 1 = 4$ depth-4 stages S. Patterns Pj and Pk are $(6, 4, 4, 2, 2, 1, 1, 3, 3, 5, 5)$ and $(3, 3, 5, 5, 7, 7, 6, 6, 4, 4, 2)$, correspondingly. Arrows ↓ and ↑ show the direction of the 2-position shifts of the respective patterns. $[j, k]$ denotes linear function $x_j \oplus x_{j+1} \oplus ... \oplus x_k$ of primary variables, that accepts the application of Phase gates to, so long as contained to within the circuit. A total of 4 Phase gate stages is required; Phase gates can be applied to the individual literals selectively in the beginning or at the end of the circuit.

is computable by the following formula, $[Pj(n - 1 - 2t + i), Pk(2t - 2 + i)] = [Pj(n - 3 - 2(t-1) + i), Pk(2(t-1) + i)]$. A simple visual explanation can be given: at each application of S pattern Pj is shifted by two positions to the left (down, Fig. 2), whereas pattern Pk gets shifted by two positions to the right (up, Fig. 2).

Observe that every $[j, k]$, $j = 1..n, k = 1..n, j \leq k$ is being generated. Indeed, a given $[j, k]$ may only be generated at most once by the 0 to m applications of the circuit S. This is because once a given j meets a given k for the first time, at each following step, the respective value k gets shifted away from j to never meet again. We next employ the counting argument to show that all functions $[j, k]$ are generated. Indeed, the total number of functions generated by 0 to m applications of the stage S is $(m+1)n = \left(\frac{n-1}{2} + 1\right) n = \frac{n(n+1)}{2}$, each linear function generated is of the type $[j, k]$ $(j = 1..n, k = 1..n, j \leq k)$, none of which can be generated more than once, and their total number is $\frac{n(n+1)}{2}$. This means that every $[j, k]$ is generated.

We illustrate the construction of the circuit implementing -$\widehat{\text{CZ}}$- for $n = 7$ in Fig. 2.

For even $n = 2m$ the construction works similarly. The patterns Pj and Pk are $(n, n - 2, n - 2, n - 4, n - 4, ..., 2, 2, 1, 1, 3, 3..., n-3, n-3, n-1)$ and $(3, 3, 5, 5, ..., n-1, n-1, n, n, n-2, n-2, ..., 4, 4, 2, 2)$, respectively. The formula for computing the linear function $[j, k]$ for i^{th} coordinate after t applications of S is $[Pj(n-2t+i), Pk(2t-2+i)]$. After m applications of the circuit S we generate linear functions $x_n, x_{n-1}, ..., x_4, x_2$ in addition to the m new linear functions of the form $[j, k]$ $(j < k)$. ∎

To consider circuit depth makes most sense when applied to measure depth across most computationally intensive operations. In both of the two leading approaches to quantum information processing, and limiting the attention to fully programmable universal quantum machines, superconducting circuits [15] and trapped ions [10], the two-qubit gates take longer to execute and are associated with lower fidelity. As such, they constitute the most expensive resource and motivate our choice to measure depth in terms of the two-qubit operations. The selection of the LNN architecture to measure the depth over is motivated by the desire to restrict arbitrary interaction patterns to a reasonable set. Both superconducting and trapped ions qubit-to-qubit connectivity patterns [10], [15] are furthermore such that they allow embedding the linear chain in them.

A further observation is that the two-qubit CNOT gate may not be native to a physical implementation, and therefore the CNOT implementation may likely use correcting single-qubit gates before and after using a specific two-qubit interaction. This means that interleaving the two-qubit gates with the single-qubit gates such as done in the proof of Theorem 6 may not increase the depth, and restricting depth calculation to just the two-qubit stages is appropriate. We did, however, report enough detail to develop depth figure over both single- and two-qubit gates for the implementations of stabilizer circuits relying on our construction.

Corollary 7: Arbitrary n-qubit stabilizer unitary can be executed in two-qubit gate depth $14n - 4$ as an $\{H, P, CNOT\}$ circuit over the LNN architecture.

Proof: Firstly, observe that -H-C-CZ-P-H-P-CZ-C = -H-

C-$\widehat{\text{CZ}}$-P-H-P-$\widehat{\text{CZ}}$-C-. This is because both -$\widehat{\text{CZ}}$- stages reverse the order of qubits, and therefore the effect of the qubit reversal cancels out. The two-qubit gate depth of the -C- stage is $5n$ [18], and the two-qubit gate depth of the -$\widehat{\text{CZ}}$- stage is $2n+2$, per Theorem 6. This means that the overall two-qubit gate depth is $14n+4$. This number can be reduced somewhat by the following two observations. Name individual stages in the target decomposition as follows, -H-C_1-$\widehat{\text{CZ}_1}$-P-H-P-$\widehat{\text{CZ}_2}$-C_2-. Using the construction in Theorem 6, we can implement -$\widehat{\text{CZ}_1}$- without the first S circuit through applying Phase gates at the end of it (see Fig. 2 for illustration). The first S circuit can then be combined with the -C_1- stage preceding it. This results in the saving of 4 layers of two-qubit computations. Similarly, -$\widehat{\text{CZ}_2}$- can be implemented up to S if it is implemented in reverse, and phases are applied in the beginning (the end, but invert the circuit). This allows to merge depth-4 computation S with the stage -C_2- that follows. These two modifications result in the improved depth figure of $14n-4$. ∎

Observe how the aggregate contribution to the depth from both -CZ- stages used in this paper, $\sim 4n$, is less than that from a single -C- stage, $5n$. The result of [18] can be applied to the 11-stage decomposition -H-C-P-C-P-C-H-P-C-P-C- of [1] to obtain a two-qubit gate depth-$25n$ LNN-executable implementation of an arbitrary stabilizer unitary. In comparison, our reduced 8-stage decomposition -H-C-CZ-P-H-P-CZ-C- allows execution in the LNN architecture in only $14n-4$ two-qubit stages.

IV. STABILIZERS AND THE SYMPLECTIC GROUP

We now establish a normal form for stabilizer circuits that eliminates 2 of the layers of the 11-layer form given in [1], while using same types of layers. As already mentioned, the stabilizer circuits form a finite group which, modulo the group that is generated by the center and the Pauli subgroup, is isomorphic to the binary symplectic group defined as follows (see also [7], [8] and [9, Chap. 2]):

Definition 8: The group $\mathrm{Sp}(2n, \mathbb{F}_2)$ of symplectic matrices of size $2n \times 2n$ with entries over the finite field $\mathbb{F}_2 = \{0, 1\}$ is defined as $\mathrm{Sp}(2n, \mathbb{F}_2) := \{A \in \mathrm{GL}(n, \mathbb{F}_2) : A^t J A = J\}$, where $J = \left[\begin{smallmatrix} \mathbf{0}_n & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{smallmatrix}\right]$, and $\mathbf{1}_n$ and $\mathbf{0}_n$ denote the identity matrix and the all zero matrix of size $n \times n$ (the subscript may furthermore be dropped when it is clear what the dimension is; $\mathbf{0}$ may furthermore be used to denote a rectangular matrix), respectively.

Similarly to [1] we can work with a tableau representation for symplectic matrices, where we omit column vector r as in [1], which corresponds to an overall sign that, if needed, can be obtained via a single layer of Z gates. Definition 8 implies that the square block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is symplectic if and only if the following four conditions hold:

$$A^t C = C^t A, \quad A^t D + C^t B = \mathbf{1}_n,$$
$$B^t D = D^t B, \quad B^t C + D^t A = \mathbf{1}_n. \tag{3}$$

In other words, two columns c_i and c_j of M are perpendicular with respect to the symplectic inner product, unless they form one out of n symplectic pairs (c_i, c_{n+i}), where $i =$

0, 1, ..., $n-1$, and in which case the symplectic inner product evaluates to 1. It should be noted that if M is symplectic, so is M^{-1}, as the symplectic matrices form a group. As $M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$, the equation $(M^{-1})^t J M^{-1} = J$ implies that the following four conditions hold for M as well:

$$AB^t = BA^t, \quad AD^t + BC^t = \mathbf{1}_n,$$
$$CD^t = DC^t, \quad CB^t + DA^t = \mathbf{1}_n. \tag{4}$$

In other words, also two rows r_i and r_j of M are perpendicular with respect to the symplectic inner product, unless they form one out of n symplectic pairs (r_i, r_{n+i}), where $i = 0, 1, ..., n-1$, and in which case the symplectic inner product evaluates to 1.

Equations (3) and (4) will be useful later when we bring a given stabilizer circuit, represented as a symplectic matrix, into a suitable normal form.

The right side action of the stabilizer circuit layers -H-, -P-, and -C- on a symplectic matrix M can be described as follows (see also [1]):

- Right multiplication with a Hadamard gate on qubit k corresponds to exchanging columns k and $n+k$ of M.
- Right multiplication with a Phase gate on qubit k corresponds to the addition modulo 2 of column k of M to column $n+k$;
- Right multiplication with a CNOT gate with control j and target k, $1 \le j, k \le n$, corresponds to the addition modulo 2 of column j to column k of M and the addition modulo 2 of column $n+k$ to column $n+j$ of M.

Similarly, the left side action on the rows of M can be defined.

V. BN-PAIRS AND BRUHAT DECOMPOSITION

A property of the symplectic group that we exploit to show an asymptotically optimal decomposition is that this group can be written as a disjoint union

$$\mathrm{Sp}(2n, \mathbb{F}_2) = \bigsqcup_{w \in W} BwB, \tag{5}$$

where B is the Borel subgroup of $\mathrm{Sp}(2n, \mathbb{F}_2)$ and W labels a system of representatives of the Weyl group of $\mathrm{Sp}(2n, \mathbb{F}_2)$. For complex Lie group this decomposition is also known as the Bruhat decomposition [5]. However, even over a finite field such as $\mathbb{F}_2$ the decomposition eq. (5) can be suitably defined using the notion of BN-pairs [3, Chap. 14], [24]. As we will see below, we can identify B with a subgroup of $\mathrm{Sp}(2n, \mathbb{F}_2)$ that is isomorphic to a subgroup of the upper triangular matrices, and we can identify W with a wreath product of $\mathbb{Z}_2$ with S_n which corresponds to the group generated by all qubit permutations together with all possible Hadamard gate combinations on n qubits.

Definition 9: (BN pair) Let G be a group and $B, N \subseteq G$ be two subgroups such that $G = \langle B, N \rangle$ and $T := B \cap N$ is a normal subgroup of N. Let S be a set of generators for $W := N/T$. Denote by $C(w) = BwB$ the double coset corresponding to the representative $w \in W$. If the following two properties hold for all $s \in S$ and all representatives $w \in W$

$$\mathcal{C}_n = \begin{pmatrix} \begin{array}{c|c} A & \\ \hline & D \end{array} \end{pmatrix}, \quad \mathcal{C}_n^{\downarrow} = \begin{pmatrix} \begin{array}{c|c} A & \\ \hline & D \end{array} \end{pmatrix}, \quad \mathcal{B}_n^0 = \begin{pmatrix} \begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & B \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array} \end{pmatrix}, \quad \mathcal{B}_n = \begin{pmatrix} \begin{array}{c|c} A & B \\ \hline & D \end{array} \end{pmatrix}$$

Fig. 3. Several important subgroups of the stabilizer group on n qubits and a visualization of the structure of the corresponding symplectic matrices: $\mathcal{C}_n$ corresponds to the group generated by all CNOT gates. The block D here is equal to $(A^{-1})^t$. If we consider only such CNOT$(x; y)$ gates where the control x and the target y labels satisfy $x \leq y$, we obtain the subgroup $\mathcal{C}_n^{\downarrow}$. In circuit terms, this means that the circuits in $\mathcal{C}_n^{\downarrow}$ may be written with only CNOT gates where the targets are lower than the controls. The circuits in $\mathcal{B}_n^0$ correspond to those that can be written in the form CPC^{-1}, where $C \in \mathcal{C}_n^{\downarrow}$ and P is a layer of Phase gates. Necessarily, B must then be symmetric, i.e., $B = B^t$. Finally, $\mathcal{B}_n$ denotes a Borel subgroup which is a maximal solvable subgroup of the stabilizer group. Again, $D = (A^{-1})^t$ holds and furthermore $AB^t = B^t A$. Note that any element $B \in \mathcal{B}_n$ can be written as $B = CP$, where $C \in \mathcal{C}_n^{\downarrow}$ and $P \in \mathcal{B}_n^0$. Note further that $\mathcal{C}_n$ and $\mathcal{B}_n$ have n^2 Boolean degrees of freedom, whereas $\mathcal{C}_n^{\downarrow}$ and $\mathcal{B}_n^0$ have $n^2/2$ Boolean degrees of freedom.

- $C(s)C(w) \subseteq C(w) \cup C(sw)$,
- $sBs^{-1} \not\subseteq B$,

then (B, N) is called a *BN-pair* and the data (G, B, N, S) is called a *Tits system*, see also [3], [5], [24].

For the group $G = \mathrm{Sp}(2n, \mathbb{F}_2)$ the subgroup B can be identified with the set $\mathcal{B}_n$ defined in Fig. 3. Completing the description of the BN-pair in case of $\mathrm{Sp}(2n, \mathbb{F}_2)$ we have to determine the subgroups N, T, and W.

In case of the finite field $\mathbb{F}_2$ it turns out that T is trivial and N consists of the group generated by all permutation matrices and all Hadamard gates. This means that a set S of generators for W can be defined as

$$S := \left\{ \begin{bmatrix} \mathbf{0}_k & & \mathbf{1}_k & \\ & \mathbf{1}_{n-k} & & \mathbf{0}_{n-k} \\ \mathbf{1}_k & & \mathbf{0}_k & \\ & \mathbf{0}_{n-k} & & \mathbf{1}_{n-k} \end{bmatrix} : k = 0..n \right\}$$
$$\cup \left\{ \begin{bmatrix} \tau_i & \\ & \tau_i \end{bmatrix} : \tau_i = (i, i+1), i = 1..n-1 \right\}. \quad (6)$$

The first set in eq. (6 corresponds to the tensor products of Hadamard matrices, namely $W = \{w_k : k = 0, 1, ..., n\}$, where $w_k = H_2^{\otimes k} \otimes \mathbf{1}_2^{\otimes n-k}$, whereas the second set corresponds to wire permutations of adjacent wires. Furthermore, we note that

$$\mathcal{B}_n = \left\{ \begin{bmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & (A^t)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \right.$$
$$\left. : A \in \mathrm{GL}(n, \mathbb{F}_2), B \in \mathbb{F}_2^{n \times n}, B = B^t \right\},$$

which implies that $\mathcal{B}_n$ is isomorphic to a subgroup of the upper triangular matrices, i.e., in particular, it is a solvable group. This decomposition also implies that there are $n^2/2$ Boolean degrees of freedom in the part corresponding to $\begin{bmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & (A^t)^{-1} \end{bmatrix}$ and $n^2/2$ Boolean degrees of freedom in the part corresponding to $\begin{bmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix}$ as B is symmetric. Hence, matrices in $\mathcal{B}_n$ have an overall of n^2 Boolean degrees of freedom.

Finally, note that the elements of the form $\mathrm{diag}(\tau, \tau)$ stabilize the set $\mathcal{B}_n^0$ as they leave the diagonal part invariant and map the set of symmetric matrices into itself.

VI. COMPUTING THE BRUHAT DECOMPOSITION

We first state two lemmas that will be useful later for a step-wise decomposition of a given stabilizer circuit.

Lemma 10 ([1]): For any symmetric matrix $A \in \mathbb{F}_2^{n \times n}$ there exist matrices $\Lambda, U \in \mathbb{F}_2^{n \times n}$ such that $A = UU^t + \Lambda$, where Λ is diagonal and U invertible and upper triangular.

Proof: In [1] a decomposition $M = LL^t + \Lambda$ was derived, where L is lower triangular. By conjugating this expression with a permutation matrix that exchanges the rows $(1, n), (2, n - 1), ...$ we see that the same proof also gives rise to a decomposition into $M = UU^t + \Lambda'$ with U upper triangular and some diagonal matrix Λ'. ∎

Corollary 11: Any matrix in $\mathcal{B}_n$ can be written in the form -C-P-C-P- or alternatively in the form -P-C-P-C- with all -C-layers consisting of gates in $\mathcal{C}_n^{\downarrow}$.

Proof: To see this, we first apply Lemma 10 decompose a given matrix $A = \begin{bmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix}$ into the product $A = \begin{bmatrix} \mathbf{1}_n & UU^t \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix} \begin{bmatrix} \mathbf{1}_n & \Lambda \\ \mathbf{0}_n & \mathbf{1}_n \end{bmatrix}$. Now, the first factor can be implemented in the form -C-P-C- and we get the following overall circuit of the form -C-P-C-P- for A:

$$A = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & (U^t)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} U^{-1} & \mathbf{0} \\ \mathbf{0} & U^t \end{bmatrix} \begin{bmatrix} \mathbf{1} & \Lambda \\ \mathbf{0} & \mathbf{1} \end{bmatrix}.$$

Clearly, the -C- layers are in $\mathcal{C}_n^{\downarrow}$. The other decomposition -P-C-P-C- is obtained similarly, by factoring out the Λ component on the left. ∎

Lemma 12: For any matrix $M \in \mathbb{F}_2^{n \times 2n}$ that is the lower $n \times 2n$ part of a $2n \times 2n$ symplectic matrix, there exist a lower triangular matrix L, an upper triangular matrix U, permutation matrices $\sigma, \tau \in S_n$, and k, $0 \leq k \leq n$, such that

$$M = L\, \sigma \begin{bmatrix} \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix} \mathrm{diag}(\tau, \tau)\, \mathrm{diag}(U, (U^{-1})^t).$$

Proof: The main idea is to use the fact that any matrix $M \in \mathbb{F}_2^{n \times 2n}$ can be decomposed into the product of a triangular matrix, a permutation pattern (i.e., a matrix that has at most one non-zero entry in each row and column), and another triangular matrix. LU decomposition with pivoting is a special case of this decomposition [12, Theorem 3.4.2], [14, Theorem 3.5.7], however, in our situation we cannot assume that we know the pivoting of the matrix. Using L, P, and U as

shorthand for lower triangular, permutation pattern, and upper triangular matrices, it is known that all four combinations $M = LPL = LPU = UPL = UPU$ are possible, see, *e.g.*, [23] for a discussion. For instance, for LPL we start in the upper right hand corner of M and eliminate going down and left. For LPU, we start in the upper left corner and eliminate going down and right. The remaining pattern defines the P-part of the matrix.

Since, by assumption, M is a part of the $2n \times 2n$ symplectic matrix, we obtain that $\text{rk}(M) = n$ which means that using an LPU decomposition on the left $n \times n$ block of M we can find L_1 and U_1 such that $L_1 M \text{diag}(U_1, (U_1^{-1})^t) = [P_1 | M_1]$, where P_1 is a permutation pattern and M_1 is another matrix. By considering the support of P_1 we can define row indices $R := \{i \in \{1, 2, ..., n\} : (P_1)_{i,*} = 0^n\}$ and column indices $C := \{j \in \{1, 2, ..., n\} : (P_1)_{*,j} = 0^n\}$. If $k := \text{rk}(P_1)$, then clearly $|R| = |C| = n-k$. Using an LPL decomposition on the restriction of the right block M_1 of this new matrix to the rows and columns in $R \times C$, we therefore obtain L_2, L_2', and permutation matrices σ, τ such that $\sigma L_2 L_1 M \text{diag}(U_1, (U_1^{-1})^t) \text{diag}(((L_2')^{-1})^t, L_2') \text{diag}(\tau, \tau) = \begin{bmatrix} \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix}$ for certain $D_1, D_2 \in \mathbb{F}_2^{n \times n}$. ∎

Theorem 13: Any Clifford circuit on n qubits can written in the form -P-C-P-C-H-C-P-C-P-.

Proof: We start with the $2n \times 2n$ symplectic matrix M of the form

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right],$$

where A, B, C, and D are in $\mathbb{F}_2^{n \times n}$. We next give an algorithm that synthesizes M in a canonical form. The algorithm proceeds in several steps, by clearing out the entries of M via left-hand and right-hand multiplications by other matrices, until finally only a permutation matrix remains, which then corresponds to a Hadamard layer up to a permutation of qubits.

Step 1. We apply Lemma 12 to the submatrix $[C|D]$. Note that since $M \in \text{Sp}(2n, \mathbb{F}_2)$ we have $C^t D = D^t C$, *i.e.*, the conditions to the lemma are satisfied and we can find a lower triangular matrix $L \in \text{GL}(n, \mathbb{F}_2)$ and a lower triangular matrix $U \in \text{GL}(n, \mathbb{F}_2)$ and two permutation matrices $\sigma, \tau \in S_n$ such that

$$\begin{bmatrix} \sigma & \mathbf{0} \\ \mathbf{0} & \sigma \end{bmatrix} \begin{bmatrix} (L^t)^{-1} & \mathbf{0} \\ \mathbf{0} & L \end{bmatrix} [C|D] \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & (U^t)^{-1} \end{bmatrix} \begin{bmatrix} \tau & \mathbf{0} \\ \mathbf{0} & \tau \end{bmatrix}$$
$$= \left[\begin{array}{cc|cc} \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{array} \right],$$

where $0 \le k \le n$ and $D_1 \in \mathbb{F}_2^{k \times k}$ and $D_2 \in \mathbb{F}_2^{k \times (n-k)}$ and $\mathbf{0}$ denotes all zero-matrices of the appropriate sizes. Application of these operations to the initial matrix forces some simplifications:

$$\left[\begin{array}{c|c} \sigma & \mathbf{0} \\ \hline \mathbf{0} & \sigma \end{array} \right] \left[\begin{array}{c|c} (L^t)^{-1} & \mathbf{0} \\ \hline \mathbf{0} & L \end{array} \right] M \left[\begin{array}{c|c} U & \mathbf{0} \\ \hline \mathbf{0} & (U^t)^{-1} \end{array} \right] \left[\begin{array}{c|c} \tau & \mathbf{0} \\ \hline \mathbf{0} & \tau \end{array} \right]$$
$$= \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{array} \right] =: M_1.$$

Here, $A_2 = \mathbf{0}$ and $A_4 = \mathbf{1}_{n-k}$ because of the symplectic condition between the last two block rows and the first two block rows of this matrix.

Step 2. We left multiply the matrix M_1 by a matrix in $\mathcal{B}_n^0$, as follows:

$$\begin{bmatrix} \mathbf{1}_k & \mathbf{0} & A_1 & A_3^t \\ \mathbf{0} & \mathbf{1}_{n-k} & A_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix} \begin{bmatrix} A_1 & \mathbf{0} & B_1 & B_2 \\ A_3 & \mathbf{1}_{n-k} & B_3 & B_4 \\ \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & B_1' & B_2' \\ \mathbf{0} & \mathbf{1}_{n-k} & B_3' & B_4' \\ \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix} =: M_2.$$

Note that since A_1 is symmetric the matrix $\begin{bmatrix} A_1 & A_3^t \\ A_3 & \mathbf{0} \end{bmatrix}$ is symmetric as well. We can apply Lemma 10 to obtain a decomposition of this upper triangular symplectic matrix applied from the left as -P-C-P-C-, where all -C- layers are in $\mathcal{C}_n^\downarrow$.

Step 3. Note that because of the symplectic condition between columns one and three of M_2 we must have $B_1' = \mathbf{1}_k$. Similarly, the symplectic condition between columns one and four of M_2 imply that $B_2' = \mathbf{0}$. Moreover, by considering the symplectic condition between rows two and three, which needs to be zero, we obtain that $B_3' = D_2^t$. We can therefore apply a final column operation to M_2 to clear out the remaining entries by multiplying on the right

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n-k} & D_2^t & B_4' \\ \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{1}_k & \mathbf{0} & D_1 & D_2 \\ \mathbf{0} & \mathbf{1}_{n-k} & D_2^t & B_4' \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n-k} & \mathbf{0} & \mathbf{0} \\ \mathbf{1}_k & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}_{n-k} \end{bmatrix} =: M_3.$$

As in Step 2, the symmetric matrix $\begin{bmatrix} D_1 & D_2 \\ D_2^t & B_4' \end{bmatrix}$ can be decomposed using Lemma 10 to obtain a representation of the overall upper triangular matrix applied from the right in the form -P-C-P-C-, where again all -C- layers are in $\mathcal{C}_n^\downarrow$.

The final matrix M_3 corresponds to a sequence of Hadamard gates applied to the first k qubits.

Overall, we applied the sequence

$$U_1 \pi_1 T_1 M T_2 \pi_2 U_2 = H,$$

where H is a product of Hadamard matrices applied to the first k basis states, $T_1, T_2 \in \mathcal{C}_n^\downarrow$, $\pi_1, \pi_2 \in S_n$, and $U_1, U_2 \in \mathcal{B}_n^0$. Multiplication by inverses from both sides yields

$$M = T_1^{-1} \pi_1^{-1} U_1^{-1} H U_2^{-1} \pi_2^{-1} T_2^{-1}.$$

Now, notice that permutations stabilize $\mathcal{B}_n^0$, *i.e.*, we can find $V_1, V_2 \in \mathcal{B}_n^0$ such that

$$M = T_1^{-1} V_1 \pi_1^{-1} H \pi_2^{-1} V_2 T_2^{-1}.$$

Note that V_1 is of the form -C-P-C-P- with the first -C- layer in $\mathcal{C}^\downarrow$, *i.e.*, T_1^{-1} and V_1 can be combined into one matrix $W_1 \in \mathcal{B}_n$. Similarly, V_2 can be written in the form -P-C-P-C-

and therefore V_2 and T_2^{-1} can be combined into one matrix $W_2 \in \mathcal{B}_n$. Note finally that we can implement $\pi_1^{-1} H \pi_2^{-1}$ using a single layer of Hadamard gates H_1 acting non-trivially on some k qubits, and merge the qubit swapping stage with either W_1 or W_2. Overall, we have that M can be written as

$$M = W_1 H_1 \pi W_2 \in \text{-C-P-C-P-H-P-C-P-C-.}$$

Since -C-P-C-P- circuit can be written as a -P-C-P-C- circuit, the claimed decomposition follows. ∎

Combining the results of Theorems 2 and 13, and Corollary 7 allows to obtain the main result of this paper,

Corollary 14: An arbitrary stabilizer circuit can be written as a 7-stage layered decomposition -C-CZ-P-H-P-CZ-C-. It is executable in the LNN architecture as a two-qubit gate depth-$(14n - 4)$ circuit.

Corollary 15: An arbitrary Clifford unitary can be implemented as a quantum circuit over the library $\{H, P, Z, CNOT, CZ\}$ using at most one Phase gate.

Proof: Consider the 7-stage decomposition -C-CZ-P-H-P-CZ-C- given by Corollary 14. Observe that the Phase gates may be required only in the -P-H-P- stage of the computation. Whenever a qubit experiences the application of more than one Phase gate, the corresponding single-qubit circuit can be rewritten using no more than one Phase gate by combining different Phase gates into one, or up to the undetectable global phase of $\frac{1+i}{\sqrt{2}}$, as follows: $PHP \mapsto HPZH$, $PHP^\dagger \mapsto HPZHZ$, $P^\dagger HP \mapsto ZHPZH$, and $P^\dagger HP^\dagger \mapsto ZHPZHZ$. The Phase gates are now contained to a single layer of at most n. Observe further that two parallel Phase gates $P(a)P(b)$ can be implemented as a circuit using one Phase gate as follows,

$$P(a)P(b) = CZ(a;b)CNOT(a;b)P(b)CNOT(a;b). \quad (7)$$

Folding the transformation in eq. (7) k times allows to implement a layer of $k+1$ Phase gates using a circuit that contains only one Phase gate. Therefore, using such folded construction allows to implement arbitrary Clifford circuit using at most one Phase gate. ∎

Corollary 16: Defining $B := \mathcal{B}_n$ and $N(= W)$ to be the group generated by -H- and all wire permutations we obtain that B and N define a BN-pair for $Sp(2n, \mathbb{F}_2)$.

Proof: From Theorem 13 we obtain, in particular, that B and N generate the entire group $Sp(2n, \mathbb{F}_2)$. Clearly we have that $T = B \cap N$ is trivial, *i.e.*, it is normal in N. The stated property $sBs^{-1} \nsubseteq B$ for all $s \in S$ clearly holds for our choice of the generator set S in eq. (6), as Hadamard as well as qubit swaps do not preserve the directional CNOT gates. Finally, to establish the coset multiplication rule $C(s)C(w) \subseteq C(w) \cup C(sw)$ we use [6, Chap. V.6]. ∎

Corollary 17: The Bruhat decomposition gives rise to an asymptotically tight parametrization of all $2^{2n^2 + O(n)}$ stabilizer circuits.

Proof: This is a direct consequence of the decomposition into layers of the form -C-P-C-P-H-P-C-P-C- proved in Theorem 13. From the proof of the theorem we see that the -C-P-C-P- and the -P-C-P-C- layers correspond to the elements of $\mathcal{B}_n$, each of which has $n^2 + o(n^2)$ Boolean degrees of freedom. This yields the claimed statement. ∎

VII. NORMAL FORM FOR STABILIZER CIRCUITS

The Bruhat decomposition eq. (5) allows us to characterize the possible block structures that stabilizer operators might have when considered as a unitary matrix of size $2^n \times 2^n$ and how they behave under multiplication.

Definition 18: Let C be a stabilizer circuit. Let $B \cdot w(C) \cdot B$ denote the unique double coset that C lies in. Then we can represent $w(C)$ by an element in $\mathbb{Z}_2^n \rtimes S_n$, or equivalently by a matrix of the form $U\pi$, where U is a tensor product of k Hadamard matrices, where $1 \leq k \leq n$ and π is a permutation matrix of n wires. By rearranging the non-identity Hadamard operators, we can represent such an element $U\pi$ in a form $\sigma(\mathbf{1}_2^{\otimes n-k} \otimes H^{\otimes k})\tau$, where $\pi = \sigma\tau$. We call (k, σ, τ) the block structure of C.

Note that whereas U and π are unique, in general σ and τ are not, as there is a degree of freedom corresponding to elements in $S_{n-k} \times S_k$. However, the collection $I \subset \mathbb{Z}_{2^n}^2$ defined as $I := \left\{ (i, j) : |C_{i,j}| = \frac{1}{\sqrt{2^k}} \right\}$ is uniquely defined by C and the corresponding block structure (k, σ, τ). As a corollary to Theorem 13 we obtain the following multiplication rule for block structures.

Corollary 19: Let C_1 and C_2 be stabilizer circuits with block structures (k_1, σ_1, τ_1) and (k_2, σ_2, τ_2), respectively. Then the block structure of $C_1 C_2$ is of the form $(m, \sigma_3, \sigma_3^{-1} \sigma_1 \tau_1 \sigma_2 \tau_2)$, where $0 \leq m \leq k_1 + k_2$ and $\sigma_3 \in S_n$.

Proof: Let $w(C_1) = U\pi$ denote the representative of C_1 in the Weyl group. Write $w(C_1)$ as a product over the generators $S = S_h \cup S_p$ where $S_h = \{h_i = \mathbf{1}_i : i \in \{1, 2, ..., n\}\}$ and $S_p = \{p_i = (i, i+1) : i \in \{1, 2, ..., n-1\}\}$. As the Weyl group is a semidirect product we can collect the factors corresponding to S_h and S_p together and write $w(C_1) = \prod_{i=1}^{k_1} h_{t_{1,i}} \prod_{j=1}^{n_1} p_{s_{1,i}}$. We similarly write $w(C_2)$ and note that as W is a semidirect product, we get $w(C_1)w(C_2) = \prod_{i=1}^{k_1} h_{t_{1,i}} \prod_{i=1}^{k_2} h_{\pi t_{2,i}} \prod_{j=1}^{n_1} p_{s_{1,i}} \prod_{j=1}^{n_2} p_{s_{2,i}}$. As there might be cancellation between the Hadamard matrices $\prod_{i=1}^{k_1} h_{t_{1,i}}$ and $\prod_{i=1}^{k_2} h_{\pi t_{2,i}}$, we obtain that the Hadamard block can have m non-trivial factors where $0 \leq m \leq k_1 + k_2$. The permutational parts multiply, *i.e.*, we conclude that the permuted block structure of the product is of the claimed form. ∎

VIII. CONCLUSION

In this paper, we reduced the 11-stage computation -H-C-P-C-P-H-P-C-P-C- [1] into the 9-stage decomposition -C-P-C-P-H-P-C-P-C- relying on the Bruhat decomposition of the symplectic group. We showed that all -C- stages in our 9-stage decomposition correspond to upper triangular matrices. This leads to an asymptotically tight parameterization of the stabilizer group, matching its number of $2^{2n^2 + O(n)}$ degrees of freedom. We then derived a 7-stage decomposition of the form -C-CZ-P-H-P-CZ-C-, that relies on the stage -CZ-, not considered by [1]. We showed evidence that the -CZ- stage is likely superior to the comparable -C- stage. Indeed, the number of the Boolean degrees of freedom in the -CZ- stage is only about a half of that in the -C- stage, two-qubit gate counts for optimal implementations of -CZ- circuits remain smaller than those for -C- circuits (see Table I), and -CZ- computations

were possible to implement in a factor of 2.5 less depth than that for -C- stage computations over LNN architecture.

We reported a two-qubit gate depth-$(14n - 4)$ implementation of stabilizer unitaries over the gate library $\{H, P, CNOT\}$, executable in the LNN architecture. This improves previous result, a depth-$25n$ circuit [1], [18] executable over LNN architecture.

Our 7-stage construction can be written in 16 different ways, by observing that -C-CZ-P- can be written in 4 different ways: -C-CZ-P-, -C-P-CZ-, -P-CZ-C-, and -CZ-P-C-. For the purpose of practical implementation we believe a holistic approach to the implementation of the 3-layer stage -P-CZ-C- may be due, where the linear reversible function $g(x_1, x_2, ..., x_n)$ is implemented by the CNOT gates such that the intermediate linear Boolean functions generated go through the set that allows implementation of the phase polynomial part.

IX. Acknowledgements

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