Algebraic Conditions for Classifying the Positional Relationships of Two Conics and Their Applications

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Abstract In many fields of computer science such as computer animation, computer graphics, computer aided geometric design and robotics, it is a common problem to detect the positional relationships of several entities. Based on generalized characteristic polynomials and projective transformations, we derive algebraic conditions for detecting the various positional relationships of two planar conics, namely, outer separation, exterior contact, intersection, interior contact and inclusion. We then apply the results to detecting the positional relationships of a cylinder (or a cone) and a quadrics. The criteria is very effective and easier to use than other known methods.

Keywords collision detection, projective transformation, generalized characteristic polynomial, positional relationship.

1 Introduction

Detecting the positional relationships of several geometric entities in a Euclidean space is a very important problem in computer animation, computer graphics, computer aided geometric design, robotics and virtual reality. Since the computational speed and accuracy are affected by the complexity of entities, in general those complex objects are replaced with simpler geometric entities which are easier to control. Because of good geometric properties and flexible control parameters, quadric surfaces(conic curves) play a fundamental role in the above scientific fields [1].

One approach to detect the positional relationships of two quadrics is to compute the intersection curves of the two quadrics, a problem well studied by many researchers [1–9]. However finding intersection curves of two surfaces is a time consuming process.

A more intuitive method is to compute the Euclidean distance of two quadrics. By the Lagrange multiplier, the distance calculation problem is reduced to the problem of solving a univariate polynomial whose degree could be as high as 24 [10]. The distance problem could also be converted to solving a system of low-degree equations with the aid of line geometry [11]. But it is also very costly if we only need to know whether two quadrics are overlapped or separated.

A recent progress on classifying the relationships of two quadrics was made by Wang et al. [12]. They presented an algebraic condition for the separation of two ellipsoids just by checking the roots of the so-called *Segre's* characteristic polynomial of two quadrics. However, they didn't classify all the possible positional relationships of two ellipsoids. The present authors followed their technique and derived the algebraic conditions for classifying all the positional relationships of two ellipses, an ellipse and a hyperbola, an ellipse and a parabola [13, 14]. In this paper, we extend the same idea from [12–14] to present the algebraic conditions for classifying the positional relationships of two arbitrary non-degenerate conics in a real affine space AR^2. The basic idea is to apply a projective trans-

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formation such that a conic is converted to a unit circle, then the technique in [13, 14] could be applied to find the positional relationships of two conics. By a parallel projection, the positional relationships of a cylinder (or cone) and a quadric in AR^3 can be converted to those of two conics.

The rest of this paper is organized as follows. Section 2 provides some notations and definitions, and summarizes the main results in [13,14]. In Section 3, we apply a real projective transformation to two conics to transform the problem of detecting the relationships of the two conics to that of a unit circle and a conic. An algorithm is outlined to detect the relationships of two non-degenerate conics. In Section 4 and 5, we derive respectively the algebraic conditions for the positional relationships of a cylinder and a quadrics, and a cone and a quadrics. The conclusions are given in Section 6.

2 Preliminaries

A conic $\mathcal{A}$ can be represented in a quadratic form $XAX^T = 0$, where A is a real symmetric 3×3 matrix and $X = (x, y, 1)$. The conic $\mathcal{A}$ is *degenerate* if $\det(A) = 0$, otherwise it is called *proper*. Let X_0 be a point in AR^2, $X_0 = (x_0, y_0, 1)$ is called *on*, *outside* and *inside* $\mathcal{A}$ if $X_0AX_0^T = 0$, $X_0AX_0^T > 0$ and $X_0AX_0^T < 0$ respectively. It is easy to see that after a real projective transformation, the positional relationship of a point and a conic doesn't change.

Given two real conics $\mathcal{A} : XAX^T = 0$ and $\mathcal{B} : XBX^T = 0$ in AR^2, their positional relationship can be classified into *outer separation, exterior contact, intersection, interior contact* and *inclusion*. It is closed related with the *generalized characteristic polynomial* of $\mathcal{A}$ and $\mathcal{B}$ which is defined as

$$f(\lambda) = \det(\lambda A + B).$$

For two quadrics $\mathcal{A} : XAX^T = 0$ and $\mathcal{B} : XBX^T = 0$ in AR^3, where A and B are real symmetric 4×4 matrices and $X = (x, y, z, 1)$, one can similarly define their generalized characteristic polynomial and the positional relationships between a point and a quadric and between two quadrics.

It is well known that the topology of two conics won't be changed by an affine transformation. So applying an appropriate affine transformation to an ellipse and a conic, the ellipse could be converted to a unit circle and the conic can be converted into the standard form. Based on this observation and the generalized characteristic polynomial, simple algebraic conditions for the positional relationships of an ellipse and a proper conic have been given in [13, 14]. We summarize the main results below.

(1) A unit circle and an ellipse

Given an ellipse $\mathcal{A} : XAX^T = 0$ and a unit circle $\mathcal{B} : XBX^T = 0$ in AR^2, where

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & x_c^2 + y_c^2 - 1 \end{pmatrix}$$

and $0 < a \le b$, $1 \le b$.

Proposition 2.1 *[13] The positional relationships of a unit circle and an ellipse are as follows:*

1. *$\mathcal{A}$ and $\mathcal{B}$ are separated iff $f(\lambda) = 0$ has two distinct positive roots.*
2. *$\mathcal{A}$ and $\mathcal{B}$ are circumscribed iff $f(\lambda) = 0$ has a positive double root.*
3. *$\mathcal{B}$ is inside $\mathcal{A}$ iff $f(\lambda) = 0$ has three distinct negative roots, two of which are not less than $-a^2$ or three roots are $-a^2, -a^2, -\frac{b^2}{a^2}$, where $a^2 > b$.*
4. *$\mathcal{A}$ and $\mathcal{B}$ have only two intersection points iff $f(\lambda) = 0$ has two imaginary roots.*
5. *$\mathcal{A}$ and $\mathcal{B}$ have four intersection points iff $f(\lambda) = 0$ has three distinct negative roots which are not greater than $-a^2$.*
6. *$\mathcal{A}$ and $\mathcal{B}$ have two intersection points and an inner tangent point iff $f(\lambda) = 0$ has a negative double root which is less than $-a^2$ and another root which is not greater than $-a^2$.*
7. *$\mathcal{A}$ and $\mathcal{B}$ have only an inner tangent point iff $f(\lambda) = 0$ has a negative double root greater than $-a^2$ or three roots are $-a^2$, where $a^2 = b$.*

8. *$\mathcal{A}$ and $\mathcal{B}$ have two inner tangent points iff the roots of $f(\lambda) = 0$ are $-a^2, -a^2, -\dfrac{b^2}{a^2}$, where $a^2 \leq b$.*

9. *$\mathcal{A}$ is as same as $\mathcal{B}$ iff the roots of $f(\lambda) = 0$ are $-1, -1, -1$ where $a = b = 1$.*

(2) A unit circle and a parabola

Given a parabola $\mathcal{A} : XAX^T = 0$ and a unit circle $\mathcal{B} : XBX^T = 0$ in AR^2, where

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & x_c^2 + y_c^2 - 1 \end{pmatrix}$$

and $a > 0$.

Proposition 2.2 *[14] The positional relationships of a unit circle and a parabola are as follows:*

1. *$\mathcal{A}$ and $\mathcal{B}$ are separated iff $f(\lambda) = 0$ has two distinct positive roots.*

2. *$\mathcal{A}$ and $\mathcal{B}$ are circumscribed iff $f(\lambda) = 0$ has a positive double root.*

3. *$\mathcal{B}$ is inside $\mathcal{A}$ iff $f(\lambda) = 0$ has three distinct negative roots, two of which are not less than $-a^2$ and one root belongs to $(-\infty, -a^2)$, or three roots are $-a^2, -a^2, -\dfrac{1}{a^2}$ when $a > 1$.*

4. *$\mathcal{A}$ and $\mathcal{B}$ have only two intersection points iff $f(\lambda) = 0$ has two imaginary roots.*

5. *$\mathcal{A}$ and $\mathcal{B}$ have four points of intersection iff $f(\lambda) = 0$ has three distinct negative roots which are not greater than $-a^2$.*

6. *$\mathcal{A}$ and $\mathcal{B}$ have two points of intersection and an inner tangent point iff $f(\lambda) = 0$ has a negative double root and three roots are not greater than $-a^2$, where $a \leq 1$.*

7. *$\mathcal{A}$ and $\mathcal{B}$ have only an inner tangent point, where $a \neq 1$ iff $f(\lambda) = 0$ has a negative double root which is greater than $-a^2$.*

8. *$\mathcal{A}$ and $\mathcal{B}$ have two inner tangent points iff the roots of $f(\lambda) = 0$ are $-a^2, -a^2, -\dfrac{1}{a^2}$, where $a < 1$.*

(3) A unit circle and a hyperbola

Given an hyperbola $\mathcal{A} : XAX^T = 0$ and a unit circle $\mathcal{B} : XBX^T = 0$ in AR^2, where

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & -1 + x_c^2 + y_c^2 \end{pmatrix}$$

and $a, b > 0$.

Proposition 2.3 *[13] The positional relationships of a unit circle and a hyperbola are as follows:*

1. *$\mathcal{A}$ and $\mathcal{B}$ are separated iff $f(\lambda) = 0$ has two distinct positive roots not greater than b^2.*

2. *$\mathcal{A}$ and $\mathcal{B}$ have only two points of intersection iff $f(\lambda) = 0$ has two imaginary roots.*

3. *$\mathcal{A}$ and $\mathcal{B}$ have only an inner tangent point iff $f(\lambda) = 0$ has a negative double root not less than $-a^2$, or three roots are $-a^2, -b, -b$ where $a^2 < b$.*

4. *$\mathcal{A}$ and $\mathcal{B}$ have only two inner tangent points iff the roots of $f(\lambda) = 0$ are $-a^2, -a^2, -\dfrac{b^2}{a^2}$, where $a^2 \leq b$.*

5. *$\mathcal{A}$ and $\mathcal{B}$ have two points of intersection and an inner tangent point iff $f(\lambda) = 0$ has a negative double root and all roots are not greater than $-a^2$. If a root is $-a^2$, the three roots are $-a^2, -b, -b$ where $a^2 < b$.*

6. *$\mathcal{A}$ and $\mathcal{B}$ have only one outer tangent point iff $f(\lambda) = 0$ has a positive double root which is root less than b^2.*

7. *$\mathcal{A}$ and $\mathcal{B}$ have two outer tangent points iff the roots of $f(\lambda) = 0$ are $b^2, b^2, -\dfrac{a^2}{b^2}$ where $b \leq 1$.*

8. *$\mathcal{A}$ and $\mathcal{B}$ have two points of intersection and an outer tangent point iff $f(\lambda) = 0$ has a positive double root not less than b^2, where $b < 1$.*

9. *$\mathcal{A}$ and $\mathcal{B}$ have four points of intersection iff $f(\lambda) = 0$ has three distinct negative roots which are not greater than $-a^2$, or $f(\lambda)$ has two distinct positive roots not less than b^2.*

10. *$\mathcal{B}$ is inside $\mathcal{A}$ iff $f(\lambda)=0$ has three distinct negative roots, two of which are not less than $-a^2$, or three roots are $-a^2, -a^2, -\frac{b^2}{a^2}$, where $a^2 > b$.*

In this paper, we will derive algebraic conditions for the positional relationships of two arbitrary conics based on projective transformations. The tricky thing is how to find a suitable projective transformation such that a conic can be mapped to an ellipse and at the same time, the positional relationship after the transformation does not change.

3 Positional relationships of two conics

In this section, we present an algorithm to detect all the positional relationships of two proper conics in AR^2.

A conic $\mathcal{A}$ in a projective plane is defined as $XAX^T=0$, where A is a symmetric 3×3 matrix and $X=(x,y,w)$ is a homogeneous coordinate. From the projective geometry point of view, a real affine transformation may be regarded as a special real projective transformation by which the line at infinity is transformed to the line at infinity. The proper conics could be classified by the status of the intersection between the conic and the line at infinity.

Definition 3.1 *[15] The line at infinity is denoted by L and a conic is denoted by $\mathcal{A}$. $\mathcal{A}$ could be classified as follows:*

1. *$\mathcal{A}$ is an ellipse if $\mathcal{A}$ and L do not have any real intersection point.*
2. *$\mathcal{A}$ is a hyperbola if $\mathcal{A}$ and L have two real points of intersection.*
3. *$\mathcal{A}$ is a parabola if $\mathcal{A}$ and L are tangent at a real point.*

We denote the real projective plane as PR^2. Given a conic $\mathcal{A}$, we choose a line L such that it doesn't have any real intersection with $\mathcal{A}$ in PR^2. We then pick a real projective transformation $\mathcal{P}$ by which L is transformed to the line at infinity. The proper conic $\mathcal{A}$ is thus mapped to an ellipse. Since the positional relationship of two conics doesn't change after a real projective transformation, the positional relationship of two conics is converted to the positional relationship of an ellipse and a conic.

To find a line L such that it doesn't intersect with a given conic $\mathcal{A}$, we proceed as follows. Let P be the matrix corresponding to projective transformation $\mathcal{P}$, then the vector $X=(x,y,1)$ is changed to $X_*=XP^{-1}$ and the conic $\mathcal{A}: XAX^T=0$ is mapped to $\mathcal{A}_*: XAX^T = X_*PAP^TX_*^T = X_*A_*X_*^T=0$, where $A_*=PAP^T$. Since A is a symmetric matrix and $\mathcal{P}$ is a congruent transformation, the matrix $P_*=PAP^T$ can be changed to a diagonal matrix. Thus we only have to choose P such that $PAP^T = diag(1,1,-1)$.

From *Bèzout*'s theorem, two conics have four points of intersection (counting multiplicity) in a complex projective plane. If two conics have a real point of intersection at infinity, then this point could be transformed to a real finite point. Let's look at an example.

Example 3.1 *Given two parabolas $\mathcal{A}: x^2-y=0$ and $\mathcal{B}: x^2+y+5=0$ which are tangent at point $(0,1,0)$. Applying a real projective transformation $\mathcal{P}$:*

$$P=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 1 & 1\end{pmatrix}$$

the two parabolas are transformed to two ellipses in a real affine plane: $\mathcal{A}_*: x^2+y^2=1$ and $\mathcal{B}_*: x^2+4(y-5/4)^2-1/4=0$, which have a real tangent point $(0,1,1)$.

Thus the algorithm should also contain a step to remove the possible intersection points of two conics at infinity. This step can be proceeded as follows.

Denote the quadratic terms of the equations of the conics $\mathcal{A}$ and $\mathcal{B}$ as F_A and F_B. It is easy to check that $\mathcal{A}$ and $\mathcal{B}$ have real intersection points at infinity if and only if the system of $F_A=0$ and $F_B=0$ must have a real solution, and if and only if

$$\begin{aligned}\mathrm{Res}(\mathrm{F_A},\mathrm{F_B}) =&\mathrm{res}(\mathrm{F_A}|_{x=1},\mathrm{F_B}|_{x=1},\mathrm{y})\times\\ &\mathrm{res}(\mathrm{F_A}|_{y=1},\mathrm{F_B}|_{y=1},\mathrm{x})=0\end{aligned}$$

and the discriminants of F_A and F_B are greater than or equal to zero. Here $\mathrm{res}(\cdot,\cdot,\cdot)$ is the resultant with respect to the last parameter.

Denoted by $\mathcal{P}$ the real projective transformation which transforms $\mathcal{A}$ to a unit circle, and the transformed conics $\mathcal{A}$ and $\mathcal{B}$ are denoted as $\mathcal{A}_*$ and $\mathcal{B}_*$. We give the criterion for classifying the real intersection points at infinity as follows:

1. If $\mathcal{A}$ and $\mathcal{B}$ are all parabolas and they have a real point of intersection at infinity, then this point is transformed to a real tangent point of $\mathcal{A}_*$ and $\mathcal{B}_*$;

2. If $\mathcal{A}$ is a parabola and $\mathcal{B}$ is a hyperbola and they have a real point of intersection at infinity, then this point is transformed to a real intersection point of $\mathcal{A}_*$ and $\mathcal{B}_*$;

3. If $\mathcal{A}$ and $\mathcal{B}$ are all hyperbolas and they have a real point of intersection at infinity, then this point is transformed to a point of intersection of $\mathcal{A}_*$ and $\mathcal{B}_*$. Especially, if the center points of $\mathcal{A}$ and $\mathcal{B}$ are the same, their real intersection point at infinity is transformed to a real tangent point of $\mathcal{A}_*$ and $\mathcal{B}_*$.

Now we outline the algorithm to classify the positional relationships of two conics.

PR-CONICS Algorithm

Input: Two real conics in AR^2: $\mathcal{A} : XAX^T = 0$ and $\mathcal{B} : XBX^T = 0$.

Output: Positional relationships of $\mathcal{A}$ and $\mathcal{B}$.

Main Steps:

Step 1: Determine the types of $\mathcal{A}$ and $\mathcal{B}$. If one of the two conics is degenerate, goto step 7; If one of the two conics is an ellipse, goto step 3; otherwise, goto the next step.

Step 2: Compute $Res(F_A, F_B)$ and record the status of the intersection points of $\mathcal{A}$ and $\mathcal{B}$ at infinity. Transform $\mathcal{A}$ to an ellipse by a projective transformation, and apply this transformation to $\mathcal{B}$. we still denote the transformed conics as $\mathcal{A}$ and $\mathcal{B}$. Goto the next step.

Step 3: Without loss of generality, Suppose $\mathcal{A}$ is an ellipse. Apply a scale transformation to $\mathcal{A}$ and $\mathcal{B}$ by which $\mathcal{A}$ is transformed to a unit circle.

Step 4: Apply a Euclidean transformation to $\mathcal{A}$ and $\mathcal{B}$ by which $\mathcal{B}$ is transformed to a standard form. Go to the next step.

Step 5: Determine the type of $\mathcal{B}$ and compute the positional relationships of $\mathcal{A}$ and $\mathcal{B}$ by the results in [13, 14]. Goto the next step.

Step 6: Remove the points transformed from the points at infinity, report the positional relationships between $\mathcal{A}$ and $\mathcal{B}$. Stop.

Step 7: Detect the positional relationships of a line and a conic directly.

The above algorithm has been implemented in **Maple**. In the following, we give an example to illustrate the above algorithm.

Example 3.2 *Given two conics as follows:*

$$\begin{aligned} \mathcal{A}: \quad XAX^T &= x^2 - 2y^2 - xy - 2x - 3y + 2 = 0 \\ \mathcal{B}: \quad XBX^T &= 2x^2 - 8y^2 - 6xy + 2x - 4y - 1 = 0 \end{aligned}$$

Detecting the positional relationship of $\mathcal{A}$ and $\mathcal{B}$.

It is easy to check that $\mathcal{A}$ and $\mathcal{B}$ are all hyperbolas. Since $\mathrm{Res}(\mathrm{F_A}, \mathrm{F_B}) = 0$ and the quadratic term of $\mathcal{A}$ is not proportional to the quadratic term of $\mathcal{B}$, there is only one real intersection point at infinity. The center points of $\mathcal{A}$ and $\mathcal{B}$ are $(5/9, -8/9)$ and $(-14/25, -1/25)$, so the real point of intersection is transformed to the real point of intersection of $\mathcal{A}_*$ and $\mathcal{B}_*$. Choose the real projective transformation $\mathcal{P}$, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & -\frac{8}{15} & \frac{3}{5} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

then $\mathcal{A}$ becomes a unit circle, $\mathcal{B}$ is transformed to $XB^*X^T = 0$, where

$$B^* = PBP^T = \begin{pmatrix} 2 & \frac{43}{15} & -\frac{4}{3} \\ \frac{43}{15} & \frac{1}{3} & \frac{7}{3} \\ -\frac{4}{3} & \frac{7}{3} & -\frac{14}{3} \end{pmatrix}$$

Choose a Euclidean transformation to transform

$\mathcal{B}$ to a standard form $X\bar{B}X^T = 0$ with

$$\bar{B} = EB^*E^T = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $\mathcal{A}$ is transformed to $X\bar{A}X^T = 0$ with

$$\bar{A} = EA^*E^T = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & x_c^2 + y_c^2 - 1 \end{pmatrix}$$

where

$$a^2 = \frac{9990(\sqrt{8021} + 35)}{2886601} \approx 0.4310795835$$
$$b^2 = \frac{9990(\sqrt{8021} - 35)}{2886601} \approx 0.1888223405$$
$$x_c = \frac{20(251 + \sqrt{8021})}{\sqrt{16042 - 50\sqrt{8021}}(\sqrt{8021} - 35)}$$
$$y_c = \frac{10(\sqrt{8021} - 519)}{\sqrt{16042 - 50\sqrt{8021}}(\sqrt{8021} + 35)}$$

The roots of the generalized characteristic equation $f(\lambda) = \det(\lambda\bar{B} + \bar{A}) = 0$ are 0.2238069488, -1.821900564 and 0.1996239269. Because there are two positive roots greater than b^2, $\bar{A}$ and $\bar{B}$ have four points of intersection. Remove a point transformed from a point at infinity, we know $\mathcal{A}$ and $\mathcal{B}$ have three real points of intersection in AR^2. The following figure shows their relationship.

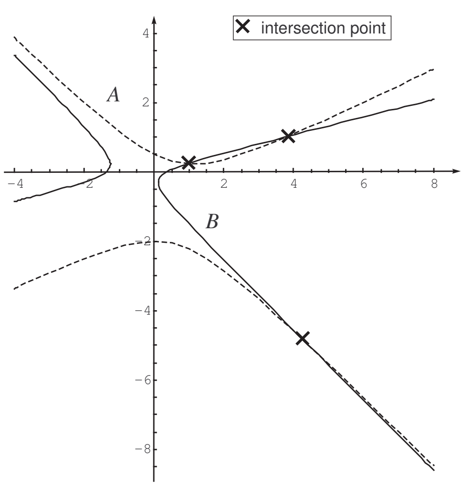

Figure 1 : Example 3.2

Example 3.3 *Given two conics as follows:*

$$\mathcal{A} : XAX^T = 25x^2 + 10xy - 155x + y^2 - 130y + 855 = 0$$
$$\mathcal{B} : XBX^T = 24x^2 - 18xy + 3y^2 - 9 = 0$$

Detecting the positional relationship of $\mathcal{A}$ and $\mathcal{B}$.

We can check that $\mathcal{A}$ and $\mathcal{B}$ are a parabola and a hyperbola. Calculate $\mathrm{Res}(\mathrm{F_A}, \mathrm{F_B}) \neq 0$, so there is no intersection point at infinity. Choose the real projective transformation $\mathcal{P}$, where

$$P = \begin{pmatrix} \frac{1}{5} & 0 & 0 \\ \frac{31\sqrt{2459}}{12295} & 0 & \frac{2\sqrt{2459}}{2459} \\ \frac{122\sqrt{2459}}{243441} & \frac{\sqrt{2459}}{99} & \frac{2\sqrt{2459}}{2459} \end{pmatrix}$$

then $\mathcal{A}$ becomes a unit circle, $\mathcal{B}$ is transformed to $XB^*X^T = 0$, where

$$B^* = PBP^T = \begin{pmatrix} 2 & \frac{43}{15} & -\frac{4}{3} \\ \frac{43}{15} & \frac{1}{3} & \frac{7}{3} \\ -\frac{4}{3} & \frac{7}{3} & -\frac{14}{3} \end{pmatrix}$$

Choose a Euclidean transformation to transform $\mathcal{B}$ to a standard form $X\bar{B}X^T = 0$ with

$$\bar{B} = EB^*E^T = \begin{pmatrix} \frac{1}{a^2} & 0 & 0 \\ 0 & -\frac{1}{b^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $\mathcal{A}$ is transformed to $X\bar{A}X^T = 0$ with

$$\bar{A} = EA^*E^T = \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ -x_c & -y_c & x_c^2 + y_c^2 - 1 \end{pmatrix}$$

where

$$a^2 = \frac{74343942895}{-521731066032 + 1928052720\sqrt{75585}} \approx 8.910732031$$
$$b^2 = \frac{74343942895}{521731066032 + 1928052720\sqrt{75585}} \approx 0.07068222725$$
$$x_c = \frac{1}{11685168} \frac{200084674290 + 4172209890\sqrt{75585}}{\sqrt{9293175750 - 15103178\sqrt{75585}}}$$
$$y_c = \frac{5}{198} \frac{\sqrt{2459}\left(-94453 + 198\sqrt{75585}\right)}{\sqrt{9293175750 - 15103178\sqrt{75585}}}$$

The roots of the generalized characteristic equation $f(\lambda) = \det(\lambda\bar{B} + \bar{A}) = 0$ are -11.38425926 and a double root 0.2352119682. Because there are two positive roots greater than b^2 and $b < 1$. $\bar{A}$ and $\bar{B}$ have two points of intersection and an outer tangent point. The following figure shows their relationship.

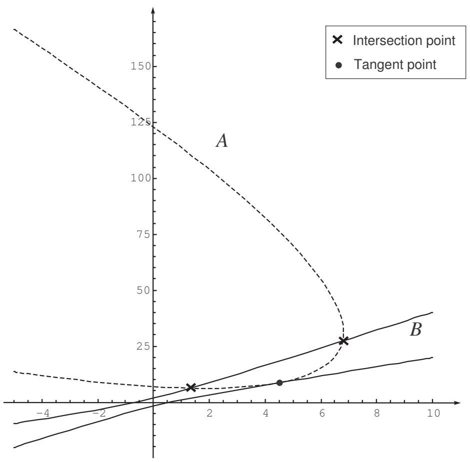

Figure 2 : Example 3.3

4 Positional relationship of a cylinder and a quadric

In an affine space AR^3, a cylinder is a special ruled surface whose generatrices have the same direction. For a cylinder $\mathcal{A}$, denote its generatrix direction as $\vec{v}$. A plane $\mathcal{F}$ which is not parallel to $\vec{v}$ intersects with $\mathcal{A}$ at a real conic curve $\mathcal{C}$. Applying a parallel projection $\mathcal{P}$ along direction $\vec{v}$, $\mathcal{A}$ is projected to $\mathcal{C}$ in $\mathcal{F}$. If there is another quadric $\mathcal{B}$, then the projection of $\mathcal{B}$ in $\mathcal{F}$ is the whole plane or an area $\mathcal{D}$ whose boundary is a conic.

We have following observation:

1. If $\mathcal{A}$ and $\mathcal{B}$ have real points of intersection, then $\mathcal{C}$ intersects with $\mathcal{D}$.
2. If $\mathcal{A}$ and $\mathcal{B}$ have real tangent points, then $\mathcal{C}$ is tangent with the boundary of $\mathcal{D}$.
3. If $\mathcal{A}$ and $\mathcal{B}$ are inclusive, then $\mathcal{C}$ and $\mathcal{D}$ are inclusive.

When $\mathcal{D}$ is not a whole plane, the converses of the above conclusions are also correct. If $\mathcal{D}$ is a plane, then $\mathcal{A}$ and $\mathcal{B}$ overlap.

Now we can describe the algorithm to detect the positional relationships of a cylinder and a quadric surface.

PR-CYL-QUAD Algorithm

1. Compute the direction $\vec{v}$ of the generatrix of $\mathcal{A}$, and choose a plane which is not parallel to $\vec{v}$ as a projective plane.
2. Compute the projective curve C_A and the boundary curve C_B of the projective area of $\mathcal{B}$.
3. Detect the positional relationship between C_A and C_B, and then derive the positional relationship of $\mathcal{A}$ and $\mathcal{B}$.

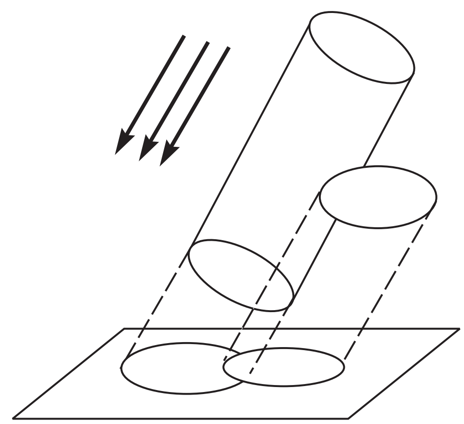

Figure 3: Sketch map of PR-CYL-QUAD algorithm

In practice we can simplify the calculation of $\vec{v}$ by applying an affine transformation. We use an example to illustrate the algorithm.

Example 4.1 *Detect the positional relationship of an elliptic cylinder*

$$\begin{aligned}\mathcal{A}: \quad XAX^T &= 12400x^2 - 25080xy - 2440xz \\ &- 21000x + 30384y^2 + 49674yz - 15900y \\ &+ 31591z^2 - 47450z - 40625 = 0\end{aligned}$$

and an ellipsoid

$$\mathcal{B}: \quad XBX^T = 25x^2 + 82y^2 - 48yz + 68z^2 - 25 = 0$$

where

$$A = \begin{pmatrix} 12400 & -12540 & -1220 & -10500 \\ -12540 & 30384 & 24837 & -7950 \\ -1220 & 24837 & 31591 & -23725 \\ -10500 & -7950 & -23725 & -40625 \end{pmatrix}$$

$$B = \begin{pmatrix} 25 & 0 & 0 & 0 \\ 0 & 82 & -24 & 0 \\ 0 & -24 & 68 & 0 \\ 0 & 0 & 0 & -25 \end{pmatrix}$$

Apply an affine transformation $\mathcal{F}$

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{627}{620} & 1 & 0 & 0 \\ -\frac{5}{4} & -\frac{4}{3} & 1 & 0 \\ \frac{9305}{4878} & \frac{7675}{7317} & 0 & 1 \end{pmatrix}$$

to $\mathcal{A}$ and $\mathcal{B}$. Denote the transformed quadrics as $\mathcal{A}^*$ and $\mathcal{B}^*$. The corresponding matrices are

$$A^* = FAF^T = \begin{pmatrix} 12400 & 0 & 0 & 0 \\ 0 & \frac{548775}{31} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{168274375}{2439} \end{pmatrix}$$

$$B^* = FBF^T = \begin{pmatrix} 25 & \frac{3135}{124} & -\frac{125}{4} & \frac{232625}{4878} \\ \frac{3135}{124} & \frac{1653961}{15376} & -\frac{245425}{1488} & \frac{243592325}{1814616} \\ -\frac{125}{4} & -\frac{245425}{1488} & \frac{45625}{144} & -\frac{35028125}{175608} \\ \frac{232625}{4878} & \frac{243592325}{1814616} & -\frac{35028125}{175608} & \frac{33448376725}{214153956} \end{pmatrix}$$

Let $z = 0$ be the projective plane, and the direction $\vec{v}$ is $(0, 0, 1)$, then the projective curve C_A of $\mathcal{A}^*$ is $37502064x^2 + 53538489y^2 - 208660225 = 0$.

Now we compute the boundary curve C_B of the projective area of $\mathcal{B}^*$. The equation of $\mathcal{B}^*$ is

$$\begin{aligned} f(x, y, z) = {} & 20580195171600x^2 + 41625104427720xy \\ & - 51450487929000xz + 78515258742000x + \\ & 88550572804929y^2 - 271553462365050yz + \\ & 221013265211100y + 260825390195625z^2 - \\ & 328406746387500z + 128575560130900 = 0 \end{aligned}$$

For every point (x, y, z) of $\mathcal{B}^*$, its normal direction is $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, where

$$\begin{aligned} \frac{\partial f}{\partial x} = {} & 20580195171600x + 20812552213860y - \\ & 25725243964500z + 39257629371000 \\ \frac{\partial f}{\partial y} = {} & 20812552213860x + 88550572804929y - \\ & 135776731182525z + 110506632605550 \\ \frac{\partial f}{\partial z} = {} & -25725243964500x - 135776731182525y + \\ & 260825390195625z - 164203373193750 \end{aligned}$$

Since ∇f of projective points are perpendicular to $\vec{v} = (0, 0, 1)$, it is obvious that $\frac{\partial f}{\partial z} = 0$.

Solve the simultaneous equations of $\frac{\partial f}{\partial z} = 0$ and $f(x, y, z) = 0$, eliminate z and we get the equation of C_B:

$$\begin{aligned} & 3048917803200x^2 + 2507980773600xy + \\ & 7794178968000x + 3019677856578y^2 + \\ & 8458537608900y + 4258477834475 = 0. \end{aligned}$$

Using the method for detecting relationship of two conics, we know C_A and C_B have two points of intersection. So $\mathcal{A}$ and $\mathcal{B}$ have intersections in AR^3.

5 A Cone and A quadric

Like cylinders, cone is also a special ruled surface. From the point of view of projective geometry, the cone and the cylinder are the same type. In affine geometry, there is only one difference between them, that is, the singular point of a cylinder is on the plane at infinity, but the singular point of a cone is a finite point. If we apply an appropriate real projective transformation to a cone, then the cone could be transformed to a cylinder in AR^2. Since the positional relationship of two quadric doesn't change after a real projective transformation, the positional relationship between a cone and a quadric is converted to the positional relationship between a cylinder and a quadric.

Denoted a cone by $\mathcal{A}$, a quadric by $\mathcal{B}$. The point O is the singular point of $\mathcal{A}$. Choose a plane $\mathcal{F}$ which passes through O. Suppose this plane and $\mathcal{A}$ have only one point in common. Apply an appropriate real projective transformation by which $\mathcal{F}$ is transformed to a plane at infinity. So $\mathcal{A}$ is transformed to a cylinder $\mathcal{A}_*$ in the new affine space. Denoted the transformed quadric $\mathcal{B}$ by $\mathcal{B}_*$. Then the problem of detecting the relationship of positions between $\mathcal{A}$ and $\mathcal{B}$ is converted to detecting the positional relationship of $\mathcal{A}_*$ and $\mathcal{B}_*$.

Since we apply a real projective transformation, the points of intersection at infinity may be transformed to finite points, and if the point O is a common point of $\mathcal{A}$ and $\mathcal{B}$, it would be transformed to a point at infinity. So we must consider these exceptional cases. However it is difficult to distinguish which points of intersection of $\mathcal{A}_*$ and $\mathcal{B}_*$ come from the points of intersection of $\mathcal{A}$ and $\mathcal{B}$ at infinity, in this paper we only concern the simple case: $\mathcal{A}$ and $\mathcal{B}$ have no real common points at infinity.

PR-CONE-QUAD Algorithm

1. Apply a congruent transformation to $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is transformed into a standard form $\mathcal{A}_1$ and $\mathcal{B}$ is transformed to $\mathcal{B}_1$.

2. Detect whether the point $(0,0,0)$ is on $\mathcal{B}_1$ or not. If it is on $\mathcal{B}_1$, $\mathcal{A}$ and $\mathcal{B}$ overlap and stop.

3. Transform $\mathcal{A}_1$ to a circular cylinder $\mathcal{A}_*$ by applying a projective transformation. $\mathcal{B}_1$ is transformed to $\mathcal{B}_*$.

4. Detect the positional relationship of $\mathcal{A}_*$ and $\mathcal{B}_*$.

We illustrate an example to demonstrate the algorithm.

Example 5.1 *Detect the positional relationship of a cone* $\mathcal{A}: \quad XAX^T = y^2 + z^2 - x^2 = 0$ *and an ellipsoid*

$$\begin{aligned}\mathcal{B}: \quad XBX^T =& 63x^2 + 79xy + 49xz + 97y^2 + 50yz \\ &+ 56z^2 - 35x - 55y - 37z - 85 = 0\end{aligned}$$

It is easy to show that the singular point $(0,0,0)$ of $\mathcal{A}$ is not on $\mathcal{B}$. Because $\mathcal{B}$ is an ellipsoid, $\mathcal{B}$ and the plane at infinity have no real points in common. Apply a real projective transformation $\mathcal{P}$ to $\mathcal{A}$ and $\mathcal{B}$, where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\mathcal{A}$ is transformed to a circular cylinder $\mathcal{A}^*$: $y^2 + z^2 = 1$ while $\mathcal{B}$ is transformed to

$$\begin{aligned}\mathcal{B}^* : 63 - 85x^2 - 55xy - 37xz - 35x+ \\ 97y^2 + 50yz + 79y + 56z^2 + 49z = 0.\end{aligned}$$

Let the plane $x = 0$ be the projective plane, the projective direction is $(1, 0, 0)$. Then the projective curve C_A of $\mathcal{A}^*$ is $C_A : y^2 + z^2 = 1$, and the boundary curve C_B of the projective area of $\mathcal{B}^*$ is

$$\begin{aligned}C_B : 36005y^2 + 21070yz + 30710y+ \\ 20409z^2 + 19250z + 22645 = 0.\end{aligned}$$

It is obvious that C_B is an imaginary ellipse, so the project area of $\mathcal{B}^*$ is the whole plane $x = 0$. It shows that $\mathcal{A}^*$ and $\mathcal{B}^*$ have real intersections, so do $\mathcal{A}$ and $\mathcal{B}$.

6 Conclusion

By applying a projective transformation and analyzing the positional relationships of a unit circle and a conic, we present algebraic conditions for all the positional relationships of two conics in AR^2. We then extend the method to detect the positional relationships of a cylinder and a quadric, a cone and a special type of quadrics. This technique transforms a three-dimensional problem to a two-dimensional problem, it reduces the computational complexity and accelerates computational speed. However, how to detect the positional relationships between two quadric surfaces is a problem worthy of further study.

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