Quantum Computing and Combinatorial Optimization

Edward Farhi
MIT

Qubits

Any two state quantum system $\begin{cases} \text{electron} \\ \text{photon} \end{cases}$

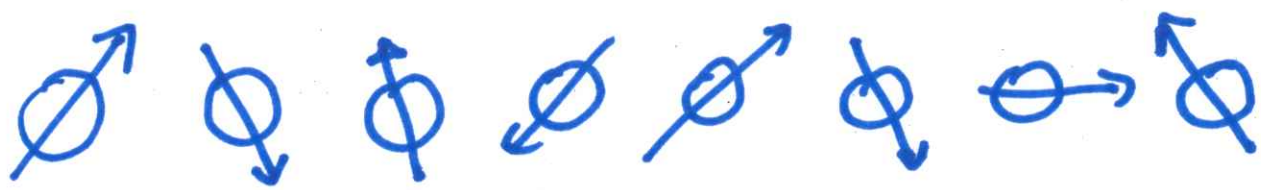
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\alpha|^2 + |\beta|^2 = 1$$

Superposition and interference!

A state of n qubits is described by 2^n complex numbers

States can be entangled and have non-classical correlations



$\begin{cases} \text{Quantum} \\ \text{Computer} \end{cases}$

Schrödinger Equation

$$i\hbar \frac{d}{dt}|\psi\rangle = H(t)|\psi\rangle$$

No violation of quantum mechanics has ever been seen!

Correctly describes

Chemistry, Materials, Light, LHC processes, Nuclear Physics ....

Quantum Computing

Seek algorithmic speedup
As a function of input size
Count the number of steps required
to reach solution

Shor factoring algorithm

d-digit number

Quantum Algorithm $O(d^2)$ steps

Best Known classical is Super Polynomial in d

Combinatorial Optimization

n bit strings $z = z_1 z_2 \ldots z_n \quad z_i = 0, 1$

Cost function

$$\boxed{C(z) = \sum_{\alpha=1}^{m} C_\alpha(z)}$$

Each C_α is a "clause" acting on a subset of the bits

$$\boxed{C_\alpha(z) = \begin{cases} 1 & z \text{ satisfies} \\ 0 & z \text{ does not} \end{cases}}$$

Goal Find z that maximizes C ← Very Hard

Goal Find z that makes $\boxed{\dfrac{C(z)}{C_{MAX}}}$ big ← Still Hard

Max E3LIN2

n variables, m equations each with 3 variables $(z_i + z_j + z_k) \bmod 2 = \begin{cases} 1 \\ 0 \end{cases}$ $z_i = 0,1$

Instance is specified by a collection of triples and a 0 or 1 for each triple

Task Find a string that maximizes the number of satisfied equations. NP hard to find the optimal solution.

Try to find a "good" solution.

Approximate Optimization

Algorithm: Guess a Random String
Achieves $\frac{1}{2}$

Limit : $\left(\frac{1}{2}+\varepsilon\right)$ would imply P=NP 2001

Bounded Occurrence : Every variable is in no more than D equations

Algorithm : $\left(\frac{1}{2}+\frac{\text{const}}{D}\right)$ 2000

Quantum Algorithm: $\left(\frac{1}{2}+\frac{\text{const}}{D^{3/4}}\right)$ 2014

Classical Algorithm : $\left(\frac{1}{2}+\frac{\text{const}}{D^{1/2}}\right)$ 2015

Quantum Approximate Optimization Algorithm

E.F., Jeffrey Goldstone, Sam Gutmann

Better Analysis: $\left(\frac{1}{2} + \frac{1}{101\, D^{1/2} \ln D}\right)$

2015

Typical: $\left(\frac{1}{2} + \frac{1}{2\sqrt{3e}\, D^{1/2}}\right)$

Can we get rid of the $\ln D$???

If $\exists$ algorithm $\left(\frac{1}{2} + \frac{\text{const}}{D^{1/2}}\right)$
for a sufficiently large constant
then $P = NP$.

Single Qubit, Easy to make

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

Like electron's spin in X-direction

n Qubits

$$|+\rangle|+\rangle \ldots |+\rangle = \frac{1}{\sqrt{2^n}} \sum_{z_1 \ldots z_n} |z_1\rangle \ldots |z_n\rangle$$

$z_i = 0, 1$

Probability of getting any string $z = z_1 \ldots z_n$ is $\frac{1}{2^n}$

just like guessing.

Cost function $C(z)$ is the number of equations satisfied by z. Easy to compute.

With a Quantum Computer easy to implement

$$U(\gamma) = e^{i\gamma C} \qquad \gamma \text{ is a real parameter}$$

$$U(\gamma)|z\rangle = e^{i\gamma C(z)}|z\rangle$$

$$|\psi_0\rangle = |+\rangle \dots |+\rangle$$

$$U(\gamma)|\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_z e^{i\gamma C(z)}|z\rangle$$

With 1 function call we change the phases of 2^n states!

Single Qubit Rotations

$$|0\rangle \rightarrow \cos\beta\,|0\rangle + i\sin\beta\,|1\rangle$$

$$|1\rangle \rightarrow i\sin\beta\,|0\rangle + \cos\beta\,|1\rangle$$

Let $V(\beta)$ rotate ~~each~~ every qubit by β.

Easy to do.

Algorithm:

Pick γ and β

Create the Quantum State

$$V(\beta)\,U(\gamma)\,|\psi_0\rangle$$

Measure in the $|z\rangle$ basis.

We showed how to pick γ and β so that the observed string z will satisfy

$$\left(\frac{1}{2} + \frac{1}{101\, D^{1/2} \ln D}\right) \cdot (\text{number of equations})$$

This works for All instances of E3LIN2.

Have provable improvement on guessing for MaxCut.

More General Quantum Algorithm

$$V(\beta_p)\, U(\gamma_p) \cdots V(\beta_2)\, U(\gamma_2)\, V(\beta_1)\, U(\gamma_1) |\psi_0\rangle$$

Pick $\gamma_1 \ldots \gamma_p, \beta_1 \ldots \beta_p$

Can only do better.

Future

Analyze higher p.
Look at other problems.
Run the QAOA on a small Quantum Computer to study its performance.
Simple Gates!
Low Circuit Depth!