Minimal Subsidies in Expense Sharing Games

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Abstract. A key solution concept in cooperative game theory is the core. The core of an expense sharing game contains stable allocations of the total cost to the participating players, such that each subset of players pays at most what it would pay if acting on its own. Unfortunately, some expense sharing games have an empty core, meaning that the total cost is too high to be divided in a stable manner. In such cases, an external entity could choose to induce stability using an external subsidy. We call the minimal subsidy required to make the core of a game non-empty the *Cost of Stability (CoS)*, adopting a recently coined term for surplus sharing games.
We provide bounds on the CoS for general, subadditive and anonymous games, discuss the special case of *Facility Games*, as well as consider the complexity of computing the CoS of the grand coalition and of coalitional structures.

1 Introduction

We begin with a motivating example. Three hospitals plan to purchase an X-ray machine. A standard machine costs \$5 million, and can fulfill the needs of up to two hospitals. An advanced machine capable of serving all three hospitals costs \$9 million. The hospital managers understand that the right thing to do is to buy the more expensive machine, which can serve all three hospitals and costs less than two standard machines, but cannot agree on how to allocate the cost of the expensive machine among the hospitals. There will always be a pair of hospitals that together need to pay at least \$6 million, and would then rather split off and buy the cheaper machine for themselves. The generous mayor solves the problem by subsidizing the expensive machine: she contributes \$3 million, and lets each hospital add \$2 million. Pairs of hospitals now have no incentive to buy the less efficient machine, as each pair together pays only \$4 million.

The example shows how external monetary funding can increase cooperation among self-interested parties. Clearly, a high enough subsidy can always induce cooperation. For example, if the mayor would decide to have the city pay for the entire expensive X-ray machine on its own, then the hospitals' (zero) costs would be irrelevant. However, we would like to consider the minimal external intervention needed to induce cooperation; in the scenario above, for example, a subsidy of \$1.5 million would suffice.

The concepts of stable payoffs, cost allocation, and subsidies have received much attention in economics, decision-making, and recently also computer science. While some papers concentrate on the fair allocation of payments, strategyproofness, and other requirements, we focus on finding the minimal subsidy that guarantees cooperation among all parties. We model situations such as the example above as Transferable Utility (TU) Expense games. In such games, every subset of agents has a fixed cost. An *imputation* is an allocation of the cost of the grand coalition containing all agents, and it is stable if no coalition can do better (i.e., pay less) on its own. The set of all stable imputations is known as the *core*, and unfortunately it may be empty (as the example above illustrates). The *Cost of Stability (CoS)* of an expense game is the minimal external payment, or subsidy, required to stabilize a game with an empty core.

Related work The term "Cost of Stability" for TU games was coined by Bachrach et al. [3, 4]. They defined it as the minimal monetary infusion required to stabilize a surplus game (where agents try to distribute a positive surplus, rather than a negative cost), focusing on computational problems in Weighted Voting games. Resnick et al. [24] extended the results to Threshold Network Flow games (suggested in [7]). The CoS in *expense games*, which are the complementary class of surplus games, has a more natural interpretation as the necessary *proportion of subsidy*, or "how much of the total expense should be subsidized?" The answer ranges between 0% (when the core is nonempty) and 100% (when no agent is willing to contribute). The relative part of the cost covered by the agents, i.e., the complement of the CoS, is called the *cost recovery ratio*.

Using different terms, several other researchers studied subsidies in expense games, and in facility games in particular (described below), sometimes adding requirements on top of the minimization of subsidies. A common assumption is that players gain some *private utility* from their participation.[1] Devanur et al. [13] suggested a mechanism that covers at least a fraction of $\frac{1}{ln(n)+1}$ in facility games, and a constant fraction of 0.462 in Metric Facility Location games—with the additional requirement of strategyproofness.

An application that has drawn much attention is routing in networks, which was initially formulated as a Minimum Spanning Tree game [12]. In this game, agents are nodes on a graph, and each edge is a connection that has a fixed price. The cost of a coalition is the price of the cheapest tree that connects all participating nodes to the source node. The CoS in this particular game is always 0, as its core is nonempty [14]. However, there is a more realistic variant of routing scenarios known as the Steiner Tree game, where nodes are allowed to route through nodes that are not part of their coalition. Meggido [20] showed that the core of the Steiner Tree game may be empty, and therefore its CoS is nontrivial. Jain and Vazirani [17] proposed a mechanism for the Steiner Tree game with a cost recovery ratio of $1/2$, under the stronger requirements of group strategyproofness.[2] Other research [25] suggested a cost sharing mechanism for Steiner Trees that does not consider strategyproofness, and showed *empirically* that it allocates at least 92% of the cost on all tested instances.

Other cost sharing mechanisms for many different games have been suggested; see Pal and Tardos [22], and Immorlica et al. [16] for an overview. Some proofs in this paper use similar techniques. Some of the proposed mechanisms pose strong requirements

[1] Our model drops this assumption, which is equivalent to assuming that participation is mandatory, or that the utility is sufficiently high to guarantee participation at any cost.

[2] To be exact, Jain and Vazirani demanded full cost recovery, and relaxed stability constraints. The bound on the CoS is achieved if we divide their proposed payments by 2.

such as group strategyproofness, in addition to stability. Therefore it is quite likely that tighter bounds on the CoS can be derived once these requirements are relaxed.

While we focus mainly on bounds, there has also been interest in the complexity of *computing* the CoS [3, 2]. Stability bounds regarding the ratio between the optimal social welfare and a core-stable social welfare were examined in affinity games [10]. External subsidies have also been suggested as a means of stabilizing normal form games [21], including a normal form version of the facility game [11].

Our contribution We analyze bounds on the CoS in the general and the anonymous case, and compare them to surplus games. We then focus on a particular class of expense games called *Facility Games* (more widely known as *Set-Cover Games*). We provide a tight bound on the CoS in facility games based on a known relation between the CoS and combinatorial properties of the Set-Cover problem, and discuss some related computational issues. We show that the bounds on facility games apply to all games whose cost function is subadditive. Interestingly, subadditivity can be further exploited to bound the subsidy even in games that are *not* subadditive. We conclude with an efficient algorithm for stabilizing coalitional structures in anonymous games.

2 Preliminaries

We denote by I the set of n agents, and by $\mathcal{S}$ the set of all possible coalitions, i.e., $\mathcal{S} = 2^I \setminus \{\emptyset\}$. $I \in \mathcal{S}$ is referred to as the *grand coalition*. An *expense game* $G = \langle I, c \rangle$ is characterized by a cost function $c : \mathcal{S} \to \mathbb{R}_+$, where $c(S)$ is the cost of coalition S. An *imputation* $\mathbf{p} \in \mathbb{R}^n_+$ is a vector whose sum is the total cost $c(I)$, which defines the amount each agent pays when the grand coalition is formed. A coalition S *blocks* imputation $\mathbf{p}$ if it can guarantee a lower payment for itself, i.e., if $c(S) < p(S) = \sum_{i \in S} p_i$. The *core* is the set of all imputations that are *stable*, i.e., not blocked by any coalition. We emphasize that the concept of the core refers only to stability of games in which only a single coalition is allowed. We discuss cases where several coalitions may exist in Section 6. For a detailed background treatment of coalitional games, see [23].

Monotonicity We say that an expense sharing game $\langle I, c \rangle$ is *monotone* if $c(S) \geq c(T)$ for all $T \subset S$. Monotonicity means that adding agents to a coalition can only increase its expenses. We will limit our attention to monotone games.

Subadditivity The game $\langle I, c \rangle$ is *subadditive*, if $c(S \cup T) \leq c(S) + c(T)$ for all S, T s.t. $S \cap T = \emptyset$. Intuitively, in subadditive expense games larger coalitions are better off, and should therefore be easier to stabilize. Like *superadditivity* in surplus sharing games [3], subadditivity means it is always best to form the grand coalition.

The cost of stability In many games, the core is empty. However an external subsidy lowers the cost of the grand coalition, thus creating a different game, possibly with a nonempty core. Formally, the *adjusted game* is an expense sharing game $G(\Delta) = \langle I, c' \rangle$ where $c'(I) = c(I) - \Delta$, and $c'(S) = c(S)$ for all $S \neq I$. A payment vector whose sum is less than $c(I)$ is referred to as a *subimputation*. Thus, imputations in the adjusted game are subimputations of the original game. Also, the imputation $\mathbf{p}$ in $G(\Delta)$

is blocked by coalition S iff the subimputation $\mathbf{p}$ in G is blocked by S. In the example given in the introduction, $c(I)$ was reduced from 9 (million) to $c'(I) = 6$, while for all $S \subsetneq I$, $c'(S) = c(S) = 5$. Naturally, we would like the external payment to be as small as possible. The *Cost of Stability* (CoS) in an expense sharing game is the minimal non-negative payment Δ s.t. the game $G(\Delta)$ has a non-empty core.[3] Thus the CoS can be formulated as the optimal solution of a linear program with exponentially many constraints, similarly to the approach in surplus sharing games; see [3] for details.

From the definition, we have that $0 \leq CoS(G) \leq c(I)$. The worst case ($CoS(G) = c(I)$) occurs when the cost of any coalition (other than I) is 0. We can derive a closed form for the Cost of Stability using *balanced collections*. Let $\delta_S \in \mathbb{R}_+$ be the coefficient of coalition S. $\boldsymbol{\delta} = \{\delta_S\}_{S \in \mathcal{S}}$ is called balanced if for every agent i, $\sum_{S: i \in S} \delta_S = 1$.

Theorem 1 (Bondareva-Shapley theorem). *The game $G = \langle I, c \rangle$ has a non-empty core iff any balanced collection of coefficients satisfies: $\sum_{S \in \mathcal{S}} \delta_S \cdot c(S) \geq c(I)$.*

For a proof and more detailed discussion, see for example [23]. By applying the theorem on the adjusted game, we can write the CoS of games with empty cores as follows:

$$CoS(G) = c(I) - \min_{\text{balanced } \boldsymbol{\delta}} \sum_{S \in \mathcal{S}} \delta_S \cdot c(S) \tag{1}$$

If the righthand-side term is negative (nonempty core), then $CoS(G) = 0$. Unfortunately, generally Equation (1) does not provide an efficient way to compute the CoS.

3 Expense Games vs. Surplus Games

We refer to transferable utility games with positive utilities as *surplus games*. In a *surplus game* $G_v = \langle I, v \rangle$, $v(S) \in \mathbb{R}_+$ is the utility that coalition S can generate, and an imputation is a division of $v(I)$ among all agents. There has been much interest in solution concepts for surplus games, as well as in their relation to expense games.

Duality The *dual* (not to be confused with linear duality) of the surplus game $G_v = \langle I, v \rangle$ is the expense game G_c defined as:

$$c(S) = \begin{cases} v(S) & , S = I \\ v(I) - v(I \setminus S) & , S \neq I \end{cases}$$

G_v's core is empty iff G_c's core is empty [9]. Duality also preserves monotonicity.

As in expense games, the CoS of a surplus game is the minimal amount that needs to be added to $v(I)$, so that the adjusted game has a non-empty core. Surplus games have already been studied [3, 24, 2], so one might ask whether expense games deserve special treatment. Further, we might conjecture that it is possible to derive the CoS of an expense game by analyzing its dual, i.e., that there is some function f such that $CoS(G_c) = f(CoS(G_v))$. Unfortunately, some important properties *are not preserved*

[3] Following [3], we define the CoS w.r.t. the *additive difference* between $c(I)$ and $c'(I)$. Related papers use the *multiplicative ratio* between the costs; the transformation is straightforward.

in the dual. For example, a game G_v can be superadditive, while its dual G_c is not subadditive nor superadditive. Furthermore, although $CoS(G_c) = 0$ implies $CoS(G_v) = 0$ (and vice versa), the following example shows that the CoS of one problem does not reveal much information about the CoS of its dual. Consider G_v, G_c s.t. $c(I) = v(I) = 1$, and the cost of all singletons in G_c is $c(\{i\}) = 0$. This means that $CoS(G_c) = 1$, as no agent has any incentive to contribute anything. This only constrains the value of coalitions of size $n-1$ in the dual game G_v to 1. $CoS(G_v)$ can still be as low as $\frac{1}{n-1}$ (if all other values are 0), or as high as $n-1$ (if $v(\{i\}) = 1$ for all agents).

4 Anonymous Games

An anonymous expense sharing game is characterized by a cost function $c : [n] \to \mathbb{R}_+$, i.e., the cost of S is $c(|S|)$. In the anonymous case, Equation (1) can be simplified:

Theorem 2. *Let G be an anonymous game. $CoS(G) = c(n) - n \cdot min_{k\leq n}\frac{c(k)}{k}$.*

Proof. We first show that there is an optimal subimputation (i.e., whose sum is minimal) in which all agents pay the same amount. Let $\mathbf{p}^*$ be an optimal subimputation, and define a new subimputation $\mathbf{q}$, with $q_j = \frac{1}{n}\sum_{i\in I} p_i^*$ for all j. For each coalition S, we have that $q(S) = \frac{|S|}{n}\sum_{i\in I} \cdot p_i^*$. Denote by $\mathcal{S}_k$ the set of all coalitions of size k; then from the stability of $\mathbf{p}^*$,

$$\forall S \in \mathcal{S}_k,\ q(S) \leq max\{p^*(S') \leq c(k) : S' \in \mathcal{S}_k\} \enspace ,$$

Thus $\mathbf{q}$ is also stable, and therefore a legal subimputation ($q(I) = p^*(I)$).

Since a coalition of size k has to pay at least $c(k)$, every agent i has to pay at least its fair share in the best possible coalition, i.e., $q_i = min_{k\leq n}\frac{c(k)}{k}$. $\square$

Without further assumptions, the CoS of an anonymous game can still reach the trivial upper bound of $c(n)$, for example if $n = 2$, $c(1) = 0;\ c(2) = 1$.

We now consider subadditivity in anonymous games, i.e., assume that $c(s+t) \leq c(s) + c(t)$. The following theorem shows that in such games the subsidy (CoS) will be approximately half of the total cost. This is similar to the corresponding result on superadditive anonymous surplus games, in which the CoS was shown to be roughly twice the value of the grand coalition [3].

Theorem 3. *Let G be a subadditive anonymous expense game.*

$CoS(G) \leq \left(\frac{1}{2} - \frac{1}{2n-2}\right) c(n)$, and this bound is tight. That is, there is a subadditive anonymous expense game for which this is exactly the CoS.

Proof. For $n \leq 2$ the theorem is trivial. Thus assume $n \geq 3$. $c(n) = c(\frac{n}{k}\cdot k) \leq \left\lceil\frac{n}{k}\right\rceil c(k)$, which means that $n\frac{c(k)}{k} \geq \frac{n}{k}\frac{1}{\lceil\frac{n}{k}\rceil}c(n)$ for any k, and in particular for $k^* = argmin\frac{c(k)}{k}$.

We denote $\frac{n}{k^*}$ by a. Note that $a \geq \frac{n}{n-1} > 1$, thus $\lceil a\rceil \geq 2$. We first look at the case $\lceil a\rceil \geq 3$. This means that $a > 2$, and thus (for $n \geq 4$)

$$\frac{a}{\lceil a\rceil} \geq \frac{a}{a+1} \geq \frac{2}{3} \geq \frac{n}{2n-2} \enspace .$$

The alternative case is $\lceil a \rceil = 2$. Here, $a = \frac{n}{n-1}$ minimizes the expression $\frac{a}{\lceil a \rceil}$ (since the denominator is fixed), and we get that $\frac{a}{\lceil a \rceil} \geq \frac{n/(n-1)}{2} = \frac{n}{2n-2}$. Note that for $n = 3$ we are either in the second case, or $k^* = 1$, and thus $\frac{a}{\lceil a \rceil} = \frac{3}{3} = 1 > \frac{n}{2n-2}$ also holds.

We showed that in any case $\frac{a}{\lceil a \rceil} \geq \frac{n}{2n-2}$, thus:

$$n \frac{c(k^*)}{k^*} \geq \frac{n}{k^*} \frac{1}{\lceil \frac{n}{k^*} \rceil} c(n) = \frac{a}{\lceil a \rceil} c(n) \geq \frac{n}{2n-2} c(n) \quad \Rightarrow$$

$$CoS(G) = c(n) - n\frac{c(k^*)}{k^*} \leq \left(1 - \frac{n}{2n-2}\right) c(n) \; = \left(\frac{1}{2} - \frac{1}{2n-2}\right) c(n) \; .$$

For tightness, consider a game where $c(n) = 2$, and $c(k) = 1$ for any $k < n$. In this game $k^* = n - 1$, and by using Theorem 2,

$$CoS(G) = c(n) - n\frac{1}{n-1} = c(n) - c(n)\left(\frac{n}{2(n-1)}\right) = c(n)\left(1 - \frac{n}{2n-2}\right) \quad \square$$

5 Facility Games

We now describe a specific domain on which we demonstrate our approach. Later, we use results for this domain to derive general results for expense games.

Facility Games (also known as Set-Cover Games) are closely related to the MinSetCover problem. In the MinSetCover problem, we are given a set $I = \{1, \ldots, n\}$, a family of subsets $F = \{A_1, \ldots, A_m\} \subseteq \mathcal{S}$, and a weight function $w : F \to \mathbb{R}_+$. We are asked to find the lightest group J s.t. $\bigcup_{j \in J} A_j = I$. We denote the the optimal set cover by $F^*(I)$ and its value by $opt(I, F) = w(F^*(I)) = \sum_{j \in F^*(I)} w_j$. We assume that each element is contained in at least one set, so $opt(I, F)$ is well-defined.

This algorithmic problem has a natural variant as an expense game: the agents are the elements, and each set represents a *facility* capable of giving service to the agents (corresponding to the elements in the set). The expense of a coalition is the minimal total price of facilities it must buy so that all of its members are served.

Formally, a facility game is a tuple $G = \langle I, F, \mathbf{w} \rangle$. The cost function is defined as $c(S) = opt(S, F|_S)$, where $F|_S = \{A_j \cap S \text{ s.t. } A_j \in F\}$. We also denote by $F^*(S) \subseteq F$ the optimal cover of S; thus $w(F^*(S)) = c(S)$.

The hospital example given in the introduction is a facility game with three agents (the hospitals). As the following lemma shows, facility games are highly expressive.

Lemma 1. *Facility games are subadditive. Furthermore, any subadditive expense game can be described as a facility game.*

Proof. We first prove subadditivity. Let $S, T \in \mathcal{S}$ be distinct coalitions; then $F^*(S) \cup F^*(T)$ is a cover of $S \cup T$. Thus $c(S \cup T) \leq w(F^*(S) \cup F^*(T)) \leq w(F^*(S)) + w(F^*(T)) = c(S) + c(T)$.

In the other direction, every subadditive game has a naïve formulation as a facility game with an exponential number of facilities: we add a facility for each coalition, whose price is the cost of the coalition. As the original game is subadditive, the cost of a coalition in the new game is exactly the price of its corresponding facility. $\square$

The CoS is tightly coupled with the key concept of the *integrality gap*. Consider the cost of the grand coalition $c(I)$. This is the optimal solution of the MinSetCover problem $\langle I, F, \mathbf{w}\rangle$, which can be written as the following integer linear program, over the variables $\{y_j\}_{A_j \in F}$:

$$\begin{aligned} &\min \sum_{A_j \in F} w_j y_j && \text{subject to:} \\ &\sum_{j: i \in A_j} y_j \geq 1 && \text{for each } i \in I, \\ &y_j \in \{0,1\} && \text{for each } A_j \in F. \end{aligned}$$

In any returned solution, the facility A_j is part of the cover $F^*(I)$ if $y_j = 1$. The linear relaxation of this program is obtained by relaxing the last condition and allowing $y_j \in [0,1]$. The difference between the optimal integer solution and the optimal fractional solution is known as the *integrality gap* of the problem.[4]

Formally, we denote by $ILP(G)$ $(= c(I))$ and $LP(G)$ the value of the optimal integer and fractional solutions of the linear program corresponding to the facility game $G = \langle I, F, \mathbf{w}\rangle$, and define the integrality gap of G as $IG(G) = ILP(G) - LP(G)$. We use the following equality, which is a known folk theorem; and also supply a simple proof, demonstrating how the optimal subimputation can be computed efficiently.

Theorem 4. *Let G be a facility game.* $CoS(G) = IG(G)$.

Proof. We define the following linear program over the variables $\{p_i\}_{i \in I}$, which is the dual program of $LP(G)$:

$$\begin{aligned} &\max \sum_{i \in I} p_i && \text{subject to:} \\ &p_i \in [0,1] && \text{for each } i \in I, \\ &\sum_{i \in A_j} p_i \leq w_j && \text{for each } A_j \in F. \end{aligned}$$

Denote by $\mathbf{p}^*$ the optimal assignment to the dual variables $\{p_i\}_{i \in I}$, and their sum (which is the optimal value of the dual) by $\hat{LP}(G)$. From strong duality we have that:

$$\sum_{i \in I} p_i^* = \hat{LP}(G) = LP(G) = \sum_{A_j \in F} w_j y_j^* \ , \tag{2}$$

where $\mathbf{y}^*$ is the optimal solution vector of the primal linear (fractional) program. We can read the dual LP as "the maximal sum of payments, such that all agents belonging to a set A_j pay at most w_j together". Consider a coalition S with cost $c(S)$. By definition

[4] We use the term "integrality gap" to denote the *difference* between the solutions, rather than their *ratio*. See also Footnote 3.

of the cost function, there is a partial cover $F'(S) = \{A_1, \ldots, A_k\}$ whose cost is $c(S) = w(F'(S)) = \sum_{r=1}^{k} w_r$. Note that:

$$\sum_{i \in S} p_i^* \leq \sum_{r=1}^{k} \sum_{i \in A_r} p_i^* \leq \sum_{r=1}^{k} w_r = c(S) \ .$$

That is, the vector $\mathbf{p}^*$ is a legal subimputation in G, as it is not blocked by any coalition S. Furthermore, any other subimputation $\mathbf{p}$ must obey the constraints of the dual program (otherwise there is a coalition $S = A_j$ that pays more than the cost of its corresponding facility), and therefore $\sum_{i \in I} p_i \geq \sum_{i \in I} p_i^*$ must hold, so it is not possible that there are better solutions (subimputations).

By definition, the CoS is the gap between the $c(I)$ and the maximal payment. Thus, from Equation (2): $CoS(G) = c(I) - \sum_{i \in I} p_i^* = IG(G)$. □

The proof also reveals the connection with the Bondareva-Shapley theorem: if all agents pay something, then due to complementary slackness, the optimal solution vector $\mathbf{y}$ is a balanced collection of coefficients (if there is an agent that pays 0, we can remove him and obtain a solution $\mathbf{y}'$).

The integrality gap of integer programs, and of MinSetCover in particular, is well-studied in the literature. We can therefore use known bounds on the IG, and apply them to the CoS. The following, for example, it due to Lovász [18]:

$$IG(G) \leq c(I) \left(1 - \frac{1}{ln(d) + 1}\right) \ , \tag{3}$$

where d is the size of the largest set in F.[5]

By joining Equation (3) and Lemma 1 to Theorem 4, we have the first part of the following corollary:

Theorem 5. *For any subadditive game $G = \langle I, c \rangle$, $CoS(G) \leq c(I) \left(1 - \frac{1}{ln(n)+1}\right)$, and this bound is tight, up to a constant.*

The tightness is due to an example by Vazirani [26], showing that the integrality gap of MinSetCover can be as high as $\frac{log_2(n)}{2}$.

It is interesting to compare this bound to the corresponding bound for superadditive surplus games, which depends on the square root of n, rather than on the logarithm [3].

6 Coalition Structures

Although the CoS of subadditive games is bounded, not all expense games are subadditive. Furthermore, in such games it is not guaranteed that forming the grand coalition is optimal (cheapest). For example, we can think of variants of facility/routing games, where there is an additional cost for using fewer facilities, or for constructing networks with a high branching factor due to increased congestion.

[5] The greedy cost-sharing scheme of [13] obtains a similar cost-recovery ratio in the worst case (and is also strategyproof), but is inferior to the result of the dual in other cases.

In such cases, the external authority may be interested in stabilizing the structure that minimizes the *social cost*, i.e., the total expenses. It is not hard to see that the same structure also minimizes the subsidy, as stability constraints for each coalition remain the same. See [3] for further discussion of this point.

Formally, a *coalition structure* is a partition of I to distinct coalitions $CS = \{T_j\}_{j=1}^k$ s.t. $\bigcup_{T_j \in CS} T_j = I$. The set $\mathcal{T}(A)$ contains all partitions of the set A, thus for $A = I$, we get the set of all coalition structures $\mathcal{CS} = \mathcal{T}(I)$. The cost of a coalition structure $CS \in \mathcal{CS}(I)$ in the game $G = \langle I, c \rangle$ is $c(CS) = \sum_{T_j \in CS} c(T_j)$. The CS-core of G (denoted by $CORE(G, CS)$) contains all subimputations such that: (a) $c(S) \geq p(S)$ for all coalitions (i.e., $\mathbf{p}$ is stable); and (b) for each $T_j \in CS$, $p(T_j) = c(T_j)$ (i.e., no transfer of payments between coalitions). In the adjusted game $G(CS, \mathbf{\Delta})$ the cost of each $T_j \in CS$ is subsidized by $\Delta_j \geq 0$, whereas the cost of any other coalition or coalition structure remains the same. As in the case of the grand coalition, $CoS(CS, G)$ is the minimal sum of $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_k)$ s.t. the CS-core of $G(CS, \mathbf{\Delta})$ is nonempty.

Finally, let $\hat{CS} \in \mathcal{CS}$ be the structure that minimizes the cost $c(\hat{CS})$. Then the minimal amount required to stabilize the cheapest structure in the game, is defined as $CoS_{CS}(G) = CoS(\hat{CS}, G)$.

Theorem 6. *For any expense game $G = \langle I, c \rangle$, $CoS_{CS}(G) \leq c(\hat{CS}) \left(1 - \frac{1}{ln(n)+1}\right)$.*

Proof. We define the *subadditive closure* of a game $G = \langle I, c \rangle$, as a coalitional game $G^* = \langle I, c^* \rangle$, whose cost function is $c^*(S) = min_{T \in \mathcal{T}(S)} c(T)$. In particular, for $S = I$ we get:

$$c^*(I) = min_{CS \in \mathcal{T}(I)} c(CS) = c(\hat{CS}) \ . \tag{4}$$

It is easy to see that $\langle I, c^* \rangle$ is a subadditive coalitional game, since any non-overlapping coalitions S_1, S_2 are a partition of the coalition $S_1 \cup S_2$.

Lemma 2 (Aumann and Dréze [1]). *Let $G = \langle I, c \rangle$ be a coalitional game, and let $CS \in \mathcal{CS}$ be a coalitional structure.*[6]

1. *If $c^*(I) = c(CS)$ then the CS-core of G is equal to the core of G^*.*
2. *Otherwise (i.e., if $c^*(I) < c(CS)$), then the CS-core of G is empty.*

Let $\mathbf{\Delta}$ be some fixed subsidy vector whose sum is $\Delta \geq 0$, and consider the optimal coalition structure $\hat{CS}$. From Equation 4 we have that $c^*(I) - \Delta = c(\hat{CS}) - \Delta$, and thus from Lemma 2 $CORE(G(\Delta)^*) = CORE(G(\hat{CS}, \mathbf{\Delta}), \hat{CS})$. In particular, Δ stabilizes $G(\Delta)^*$ iff $\mathbf{\Delta}$ stabilizes $G(\hat{CS}, \mathbf{\Delta})$. Finally, from Theorem 5,

$$\begin{aligned} CoS_{CS}(G) = CoS(G, \hat{CS}) = CoS(G^*) &\leq \left(1 - \frac{1}{ln(n)+1}\right) c^*(I) \\ &= \left(1 - \frac{1}{ln(n)+1}\right) c(\hat{CS}), \qquad \text{(from Equation (4))} \end{aligned}$$

since G^* is a subadditive expense game. □

[6] Aumann and Dréze treated superadditive *surplus games*. However, a slight variation of their work proves the lemma.

Our final result in this section shows that the subadditivity condition in Theorem 5 can in fact be weakened. Since $\{I\}$ is also a coalition structure, we get the following result for non-subadditive games as a corollary of Theorem 6.

Theorem 7. *If $c(I) \leq c(CS)$ for all $CS \in \mathcal{CS}$, then $CoS(G) \leq c(I)\left(1 - \frac{1}{ln(n)+1}\right)$.*

7 Some Notes on Computational Complexity

In some seemingly simple TU games, such as weighted voting games and threshold network flow games, finding (or even testing) if a subimputation is stable proved to be $\mathcal{NP}$-hard, while computing the value of a coalition was trivial. In particular, computing the CoS was hard in these games [3, 24].

Facility games present a situation opposite to that in the above mentioned surplus games: it is $\mathcal{NP}$-hard to compute the cost of a coalition, and in particular to know if some set of facilities $F' \subseteq F$ is optimal for the grand coalition. This follows directly from the hardness of the MinSetCover problem [26]. However, the optimal subimputation $\mathbf{p}^*$ can be computed efficiently by solving $\hat{LP}(G)$.

The computational complexity results regarding the CoS lead to an interesting trade-off between the computational power of the center and the size of the subsidy. Since the maximal costs the agents can safely pay ($\mathbf{p}^*$) do not depend on the quality of the selected set of facilities, faster computers can assist the city council in finding cheaper solutions (better F') and thereby save money on subsidies (i.e., lower $w(F') - \sum_i p_i^*$).

It is important to note that the runtime of the mechanism would be polynomial in the *description size* of the game, i.e., in the number of facilities. Therefore we cannot efficiently compute optimal payments for arbitrary subadditive expense games with the dual method, as the linear program might contain an exponential number of constraints.

The optimal coalition structure in anonymous games Recall that for anonymous games the characteristic function is given by $c : [n] \to \mathbb{R}_+$, and for every coalition structure, $c(CS) = \sum_{C \in CS} c(|C|)$. Computing the CoS of a given coalition structure is easy. As in general games, $\mathbf{p}^*$ is easy to compute. In the anonymous case, it has the form $p_i^* = min_{j \leq n} \frac{c(j)}{j}$ for all i. Thus for any *given* CS we can compute $CoS(G, CS)$ as

$$CoS(G, \hat{CS}) = c(CS) - n \cdot min_{j \leq n} \frac{c(j)}{j} \quad . \tag{5}$$

The proof is the same as the proof of Theorem 2, which is a special case for $CS = \{I\}$.

However, finding the *optimal* coalition structure might be difficult. That is, we know how much money each agent should pay in total, but we do not know how much they can make by themselves, and therefore we are not sure how much to subsidize.

Proposition 1. *Computing $CoS_{CS}(G)$ for anonymous games is in $\mathcal{P}$.*

We note here that Proposition 1 also holds for surplus sharing games, as defined in [3].

Proof. From Equation (5) it is sufficient to find the optimal coalition structure $\hat{CS}$.

Our key observation is that the problem of finding $\hat{CS}$ in an anonymous game is equivalent to solving KNAPSACK with bounded weights.

In the KNAPSACK problem, we are given n pairs $\langle w_i, x_i \rangle$, and a threshold t. We can select any pair $a_i \in \mathbb{N}$ times, in order to minimize the total weight of the sack $\sum_i a_i \cdot w_i$, while maintaining the total value above the threshold, i.e., $\sum_i a_i \cdot x_i \geq t$.[7] While the general KNAPSACK problem is $\mathcal{NP}$-hard, it can be solved by a simple dynamic algorithm, provided that either the weights (w_i) or the values (x_i) are polynomially bounded (see e.g., [19]).

Consider an anonymous cost game, with a characteristic function c. We construct a KNAPSACK instance KN with the pairs $\{\langle c(i), i \rangle\}_{i=1}^{n}$, and a threshold $t = n$ (hence the values are bounded). As $w_i = c(i)$, and $x_i = i$, we have that

$$CoS_{CS}(G) = min \sum_{i \leq n} a_i \cdot w_i \quad s.t. \quad \sum_{i \leq n} a_i \cdot x_i = n = t \ ,$$

which is the optimal knapsack solution.

Thus for anonymous coalitional games we have a dynamic algorithm that finds $\hat{CS}$ and computes $CoS_{CS}(G)$ efficiently. □

8 Discussion

Our study of minimal subsidies in expense sharing games joins together two lines of work: the ongoing study of cost sharing mechanisms with different requirements for specific game classes, and the clean formulation of bounds on the Cost of Stability that depend on the cost function's properties.

We have focused on the subadditivity property, and were able to provide tight bounds on the CoS for broad families of games, even in the absence of an efficient cost sharing mechanism. Our work also complements previous work on the Cost of Stability, highlighting the similarities and differences between cost sharing games and surplus games in terms of the magnitude of the required subsidy for achieving stability.

Several issues remain open for future research. First, better bounds on the CoS should be developed for specific classes of games, such as those suggested in [15, 6, 8, 10, 5, 2]. It is particularly interesting if dropping all requirements except stability (such as strategyproofness or computational efficiency) can result in mechanisms that are better than the existing cost sharing mechanisms described in the related work section.

Another question for future research is whether the CoS of more coalitional games, other than Set-cover and Steiner-tree games, can also be derived from the integrality gap of their underlying combinatorial problem (when there is one).

Finally, we plan to further investigate the relation between the CoS and other solution concepts, such as the Shapley Value, the nucleolus, and the least core.

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[7] The KNAPSACK problem is typically formulated as maximization of the value, rather than minimization of the weights (and this version is in fact used in the symmetric proof for surplus games). However, the problems are clearly computationally equivalent.

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