The Weighted Proportional Allocation Mechanism

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Abstract–We consider a weighted proportional allocation of resources that allows providers to discriminate usage of resources by users. This framework is a generalization of well-known proportional allocation accommodating allocation of resources proportional to weighted bids or proportional to submitted bids but with weighted payments.

We study a competition game where *everyone is selfish*: providers choose discrimination weights aiming at maximizing their individual *revenues* while users choose their bids aiming at maximizing their individual payoffs. We analyze revenue and social welfare of this game. We find that the revenue is lower bounded by $k/(k+1)$ times the revenue under standard price discrimination scheme, where a set of k users is excluded. For users with linear utility functions, we find that the social welfare is at least $1/(1+2/\sqrt{3})$ of the maximum social welfare (approx. 46%) and that this bound is tight. We extend the efficiency result to a broad class of utility functions and to multiple competing providers. We also describe an algorithm used by the provider to adjust the user discrimination weights without a prior knowledge of user utility functions and establish convergence to equilibrium points of our game.

Our results show that, in many cases, weighted proportional sharing achieves competitive revenue and social welfare, despite the fact that everyone is selfish. The mechanism allows for resource constraints described by general polyhedrons, thus accommodating a variety of resources, including bandwidth of communication networks, systems of computing resources, and sponsored search ad slots.

1 Introduction

Auctions-based Resource Allocations. Provisioning of computer systems and services using *a pay-per-use pricing* has proliferated over recent years. For example, major providers of cloud computing services use either fixed prices or auctions to sell compute instances. The two forms of sales have their own advantages and disadvantages. While fixed price schemes are simple to implement, in many scenarios they are not robust and not very flexible. For instance when the users' demands are inelastic, a small change in prices can translate to a dramatic change in the demands that can cause congestion and system failure. Furthermore, in order to update prices, providers usually need to gather enough data from sales, this inflexibility can cause inefficiency for the systems and result in low revenues. In recent years, using auction-based schemes for allocating and selling resources in computing systems has become more popular. Sales using auctions are known to be more flexible and can extract information from users faster. The fact that both users and providers can adjust their bids and parameters in auctions in a more dynamic way reflects in the tremendous improvement of the system efficiency or the revenue obtained by the providers. Moreover, the disadvantages of the auction-based approaches such as, the difficulties for users to find optimal strategies and for providers to change their platforms, have been greatly improved in recent years with availability of software that helps users to optimize the bidding strategies and several online services offering auction platforms.

Examples of using auctions for resource allocation include selling of Amazon EC2 spot instances [1]; selling of *sponsored search ad slots* to advertisers by major providers of online services and are of much interest widely across engineering systems, including *electricity markets* [26] that have been gaining momentum due to a pressing need to accommodate renewable energy sources. Furthermore, numerous auction-based proposals for allocation of system resources have been made such as allocation of disk I/O in storage systems [7] and allocation of computational resources [4] and it was even showed that sharing of network resources in the Internet by TCP connections may be seen as an auction [13, 11].

Discriminative Schemes in Auctions-based Allocations. It is well known that in practice with fixed price schemes providers usually apply different prices to different users for selling identical goods or services. This scheme, often called price discrimination, is very common in practice. It is not surprising that similar discriminative schemes are also used extensively in auction-based allocations. There are two main reasons for such a discriminative framework. First, different users require different subsets of resources owned by the provider (e.g. transmission rate for different paths through a network), thus, by discriminating among users the provider can improve the efficiency of the system. Second, different users have different valuations of the amount of resource received (e.g. different valuation of transmission rates); in this case, the providers once having learned this information can try to take advantage and use discrimination to improve the revenue.

The most well known example of auctions using such a discrimination scheme is the generalized second price

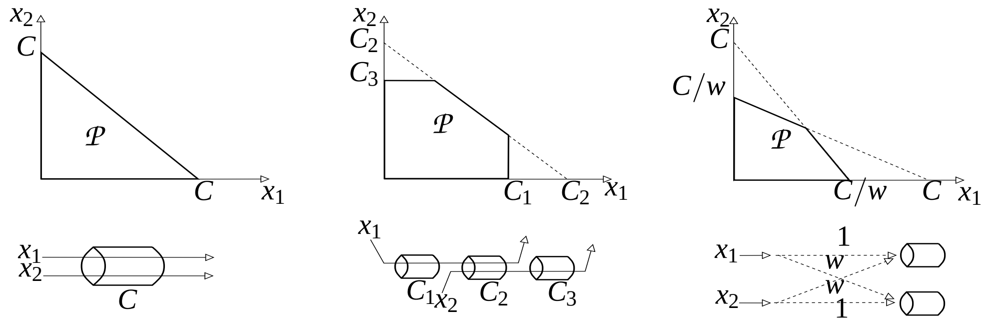

Figure 1: Examples of polyhedron constraints: (left) single link, (middle) a network of links, (right) assignment costs of a workflow to machines.

auction currently used by search engines to allocate ads. In this mechanism, the providers assigns different weights to different users (advertisers) and the mechanism is run based on weighted bids.

The Framework. In this paper, we consider a class of auctions that allows providers to discriminate among different users. Specifically, we are interested in auctions that are simple in terms of the information provided by users, and are easy to describe to users. We consider two natural instances of weighted proportional allocation: (1) *weighted bid auction* where the allocation to a user is proportional to the bid submitted by this user weighted with a discrimination weight that is selected by the provider, and the payment by the user is equal to his own bid, and (2) *weighted payment auction* where the allocation to a user is proportional the bid of this user and the payment is equal to the weighted bid, where the weight is selected by the provider. The weighted bid auction is a novel proposal while weighted payment auction was recently proposed by Ma et al [18].

As in the network pricing literature [20], we consider these allocation problems in the *full information* setting. The justification for this setting is the fact that in practice allocation auctions are run repeatedly and thus, providers can learn about the behavior and private information of users. As discussed in the beginning of this section, even in this setting there are several advantages of using proportional sharing-like auctions over fixed price schemes. Namely, both of the auctions that we consider are akin and natural generalizations of well-known proportional allocation (e.g. [13, 11, 8], see related work discussed later in this section). Thus, this class of mechanisms inherits many natural properties of the traditional proportional sharing rule making it *easy and robust to implement* in practice. First, these mechanisms are simple for bidders, they only need to know the total of others' bids. Second the allocation is a natural and continuous function of the bid vector, and, therefore, it can be robustly implemented in a distributed way (as will be shown later in our paper). From an engineering point of view, this is an important feature of practical allocation rules. For example, when users demands are inelastic (users' utilities are close to linear) proportional sharing-like mechanisms are much preferred to fixed price schemes.

Another important reason that motivates us to study these weighted proportional rules is the fact that in settings where providers' goal is to *maximize revenue*, the weighted proportional sharing is preferred over the traditional proportional sharing. As will be shown later, while weighted proportional sharing always generates near-optimal revenue, the revenue of traditional proportional sharing provides no such guarantee, and in fact, can be arbitrarily bad.

We study these allocation rules in *general convex environments* that capture many special cases of resource allocation problems such as the network bandwidth sharing, sponsored search, and scheduling of resources in cloud computing (see Figure 1 for an illustration). We provide a deeper discussion of these applications in Appendix A. We consider a provider that offers a resource to a set of users $U = \{1, 2, \ldots, n\}$ where $n \geq 1$ (for the case of multiple providers, the auctions as described in the following are applied by each individual provider). We denote with $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{q} = (q_1, q_2, \ldots, q_n)$ the vector of allocations and payments by users, respectively. The resource owned by the provider is allowed to be an arbitrary infinitely divisible resource with constraints specified by a convex set, say $\mathcal{P} \in \mathbb{R}_+^n$. An allocation vector $\vec{x}$ is said to be *feasible* if $\vec{x} \in \mathcal{P}$. The provider discriminates users by assigning a *discrimination weight* $C_i \geq 0$, for every user i. Each user i, submits a bid $w_i \geq 0$.

Our weighted bid auction determines the allocation and payment for each user as follows:

WEIGHTED BID AUCTION

For user i with bid w_i:

Allocation $\quad x_i = C_i \frac{w_i}{\sum_{j \in U} w_j}$

Payment $\quad q_i = w_i$

where discrimination weights $\vec{C}$ are chosen such that $\vec{x}$ is feasible. We may interpret the discrimination weight C_i as the maximum allocation determined by the provider for user i and x_i is the actual allocation set to a fraction $w_i / \sum_{j \in U} w_j$ of this user-specific maximum allocation.

On the other hand, the weighted payment auction determines the allocation and payment by each user as given in the following:

WEIGHTED PAYMENT AUCTION

For user i with bid w_i:

Allocation $\quad x_i = C \frac{w_i}{\sum_{j \in U} w_j}$

Payment $\quad q_i = C_i w_i$

where C is the maximum allocation to a user and the discrimination weight C_i determines the payment by user i.

Compared with the traditional proportional allocation, the weighted bid auction is more suitable for general convex resource constraints. This is not the case for weighted payment auction; while the relative allocation across users can be arbitrary by appropriate choice of user bids, the implicit assumption of the allocation rule is that $\sum_i x_i = C$, i.e. the provider is required to a priori commit to allocating a total amount of resource $C > 0$. While this is not restrictive for allocating an infinitely divisible resource of capacity $C > 0$, this allocation rule cannot accommodate more general polyhedral constraints. Thus, in this paper we will mainly focus on the weighted bid auction but also consider some properties of the weighted payment one as it is an alternative auction that allows for user discrimination, though for special type of resource constraints.

Questions Studied in this Paper. We consider a general competitive setting with multiple providers and users where *everyone is selfish*: each provider aims at maximizing own revenue and each user aims at maximizing own payoff. Ideally, an allocation mechanism would guarantee high *revenue* to a provider and high *efficiency* where by efficiency we mean social welfare (i.e. the total utility across all users) compared with best possible social welfare. In a competitive setting where everyone is selfish, it is rather unclear whether the two goals could be simultaneously achieved. Intuitively, one would expect that selfishness of users and providers may well result in either poor revenue or efficiency. For example, providers that aim at maximizing their revenue may well have an incentive to misreport availability of their resources[1]. Our main questions in this paper are:

Q1: *How much revenue can a provider obtain using weighted proportional sharing rules?*

Q2: *What is the efficiency loss in the complex system where everyone is selfish?*

Overview of our Results. Our results can be summarized in the following points:

• *Revenue*: We show that the revenue of weighted bid auction is at least $k/(k+1)$ times the revenue under standard price discrimination scheme with a set of k users excluded, which we describe in more detail in Section 3. The comparison of the revenue of a mechanism with maximum revenue obtained under another optimal pricing scheme where some users are excluded is standard in the mechanism design literature (e.g. [9]) and our revenue comparison result is novel and of general interest. The result enables us to understand conditions under which the revenue of weighted bid auction is competitive to that under the benchmark pricing scheme, e.g. the case of many users.

[1] A famous example of such a market manipulation is *California electricity crises* where, in 2000 and 2001, there was a shortage of electricity because energy traders were gratuitously taking their plants offline at peak demand in order to sell at higher prices [27].

• *Efficiency*: We establish that for linear user utility functions, weighed bid auction guarantees the social welfare of at least $1/(1+2/\sqrt{3}) \approx 0.46$ times the maximum social welfare (price of anarchy), and this bound is tight. We also show a tight efficiency bound of $1/2$ for the weighed payment auctions. We extended our efficiency result of the weighed bid auction to a broad class of utility functions, we call δ-utility functions, where $\delta \geq 0$. We show that many utility functions found in literature are δ-utility functions, in many cases with $\delta \leq 2$, and show that this class of utility functions is closed to addition and multiplication with a positive constant. We then show that if the utility functions are δ-utility functions, then the social welfare is at least $1/(1+2/\sqrt{3}+\delta)$ times the maximum social welfare, and establish that this guarantee holds for a system of multiple competing providers. A similar extension for the weighted payment auction is also shown.

• *Convergence and distributed algorithms*: We demonstrate how a provider can adjust discrimination weights in an online fashion so that allocations and payments converge to Nash equilibrium points of the competition game that we study. This shows that the information about user utility functions needed by the provider can be estimated from the bids submitted by users in an online fashion. We describe a distributed iterative scheme and the estimation needed, and prove convergence to Nash equilibrium for the case of linear user utility functions.

Finally, we note that our results also provide insights on the performance of systems where providers would claim to use traditional proportional allocation but then strategically manipulate the information signaled to users with the aim to earn higher revenue.

Related Work. The concept of allocating resources proportional to user-specific weights has a long and rich history in the context of computer systems and services. For example, it underlies the objective of generalized processor sharing [21, 2], sharing of bandwidth in communication networks [13], has been considered for allocation of storage [7] and compute instances [4]. Our weighted bid auction could be seen as a generalization of traditional proportional allocation that allows to discriminate users and accommodates general convex resource constraints.

Previous work primarily focused on analyzing social efficiency of proportional allocation in competitive environments where only users are assumed to be selfish. Kelly [13] established that under price taking users that submit a single bid for a set of resources of interest (e.g. allocation of bandwidth at each link along a path of a network connection), the proportional allocation guarantees 100% efficiency. In a subsequent work, Johari and Tsitsiklis [11] showed that the social efficiency is at least 75% under price anticipating users and assumption that each user submits an individual bid for each individual resource of interest (e.g. a bid submitted for each link along a path of a network connection). The latter result was extended by Nguyen and Tardos [19] for more general polyhedral constraints. Finally, Yang and Hajek [8], in turn, showed that the worst-case efficiency is 0% if users are restricted to submitting a single bid for a set of resources of interest. Our work stands in contrast to this line of work in that we consider a competitive environment where everyone is selfish.

The weighted payment auction studied in this paper was first proposed by Ma et al [18] where they focused on maximizing social welfare. We provide revenue and efficiency results for weighted payment auctions in environments where everyone is selfish.

Our setting of multiple providers bears quite similarity with that found in the context of ISP multihoming (e.g. [6, 22, 10]) and to that of multi-path congestion control, e.g. [5], and may also inform about competition in time-varying markets using the approach in [15].

Outline of the Paper. Section 2 introduces the resource competition game more precisely. Section 3 presents our main revenue comparison result (Theorem 1). In Section 4, we present our results on the efficiency guarantees for the case of a single provider and users with linear utility functions (Theorem 2). Section 5 extends the efficiency result to more general class of user utility functions and more general setting of multiple providers (Theorem 4). We discuss convergence to Nash equilibrium points and distributed schemes in Section 6. In Section 7, we conclude. Some of the proofs are presented in Appendix.

2 The Resource Allocation Game where Everyone is Selfish

We consider a system of n users competing for resources of a single provider; we introduce the setting with multiple providers later in Section 5.1. Recall that $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{q} = (q_1, q_2, \ldots, q_n)$ denote the vector of allocations and payments by users, respectively. The allocation vector $\vec{x}$ is feasible only if $\vec{x} \in \mathcal{P}$ where $\mathcal{P}$ is a set in R^n_+. We assume that $\mathcal{P}$ is a convex set of the form $\mathcal{P} = \{\vec{x} \in \mathbb{R}^n_+ : A\vec{x} \leq \vec{b}\}$ for some matrix A and a vector $\vec{b}$ with non-negative elements. (Note that A and $\vec{b}$ can have arbitrarily many rows, so we refer to "convex" constraints, instead of to polyhedrons.)

Suppose that $U_i(x_i)$ is the utility of allocation x_i to user i. Throughout this paper we assume that for every i, $U_i(x)$ is a non-negative, non-decreasing and continuously differentiable concave function. The payoffs for the provider and users are defined as follows. The payoff of the provider is equal to the revenue, i.e. the total payments received from all users, $R = \sum_i q_i$. On the one hand, the payoff for a user i is equal to the utility minus the payment, i.e. $U_i(x_i) - q_i$ for allocation x_i and payment q_i for user i.

The competition game that we study can be seen as the following *two-stage Stackelberg game*: in the first stage, the provider announces the discrimination weights $\vec{C}$ and then, in the second stage, users adjust their bids in a selfish way aiming at maximizing their individual payoffs. In the first stage, the provider anticipates how users would react to given discrimination weights $\vec{C}$ and sets them in a selfish way aiming at maximizing own revenue. In dynamic setting, the two stages of this game would alternate over time (we discuss this in Section 6).

In the reminder of this section we characterize Nash equilibrium for weighted-bid and weighted-payment auction.

Equilibrium of Weighted Bid Auction. We show a relation between the revenue gain and the allocation of an outcome. Given a discrimination weight C_i and the sum of the bids $\sum_j w_j$, each user i selects a bid w_i that maximizes his surplus, i.e. solves

$$\textbf{USER:} \quad \max_{w_i \geq 0} U_i\left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C_i\right) - w_i. \tag{1}$$

Under the assumed behavior of users, one can analyze the Nash equilibrium of the game. It turns out that a Nash equilibrium exists and is unique, and at Nash equilibrium the relation between revenue and allocation is captured by an implicit function.

For the provider, he would like to choose the rate to maximize the revenue. We first show the following relation between the revenue and the allocation vector $\vec{x}$.

Lemma 1 *Given discrimination weights $\vec{C}$, there is a unique allocation $\vec{x}$ corresponding to the unique Nash equilibrium. Conversely, given an allocation $\vec{x}$, there is a weight $\vec{C}$ such that $\vec{x}$ is the corresponding outcome. Furthermore, the corresponding revenue $R(\vec{x})$ is a function of $\vec{x}$ given by*

$$\sum_i \frac{U_i'(x_i)x_i}{U_i'(x_i)x_i + R(\vec{x})} = 1. \tag{2}$$

Given this result, we obtain the following optimization problem of the provider.

$$\textbf{PROVIDER:} \quad \text{maximize } R(\vec{x}) \text{ over } \vec{x} \in \mathcal{P}. \tag{3}$$

In the rest of this section we provide a proof of Lemma 1.

of Lemma 1 We have

$$x_i = C_i \frac{w_i}{\sum_j w_j} \tag{4}$$

and USER problem can be written as:

$$\text{maximize } U_i\left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C_i\right) - w_i \text{ over } w_i \geq 0. \tag{5}$$

Note that the objective function in (5) is concave in w_i, hence, at an optimum solution either $w_i = 0$ or the derivative of the objective function is zero. Setting the derivative to zero is equivalent to:

$$U_i'(x_i) \cdot C_i \frac{\sum_{j \neq i} w_j}{(\sum_j w_j)^2} = 1, \text{ for } x_i > 0.$$

It follows

$$U_i'(x_i) = \frac{(\sum_j w_j)^2}{C_i \sum_{j \neq i} w_j} = \frac{R^2}{C_i(R - w_i)} \tag{6}$$

where recall the revenue is equal to the sum of the payments by individual users, i.e. $R = \sum_j w_j$. Combining with $w_i = x_i R / C_i$ that follows from (4), we have

$$U_i'(x_i) = \frac{R}{C_i - x_i} \Leftrightarrow C_i U_i'(x_i)(1 - \frac{x_i}{C_i}) = R. \tag{7}$$

Now, $\sum \frac{x_i}{C_i} = 1$, thus, condition (7) is exactly the condition for maximizing

$$\sum_i \int_0^{x_i} C_i U_i'(t_i) \left(1 - \frac{t_i}{C_i}\right) dt_i$$
$$\text{over } \vec{x} \in \mathbb{R}_+^n \text{ and subject to } \sum \frac{x_i}{C_i} = 1.$$

Since $\int_0^{x_i} C_i U_i'(t_i) \left(1 - \frac{t_i}{C_i}\right) dt_i$ is a strictly concave function with respect to x_i, there exists a unique Nash equilibrium.

It remains to show that for an equilibrium allocation $\vec{x}$, the revenue R is given by

$$\sum_i \frac{U_i'(x_i) x_i}{U_i'(x_i) x_i + R} = 1. \tag{8}$$

From (7), we have

$$U_i'(x_i) = \frac{R}{C_i - x_i} = \frac{R}{x_i(C_i/x_i - 1)} \Rightarrow \frac{C_i}{x_i} - 1 = \frac{R}{U_i'(x_i) x_i}$$
$$\Rightarrow \frac{x_i}{C_i} = \frac{U_i'(x_i) x_i}{U_i'(x_i) x_i + R}.$$

Combining with $\sum_i x_i / C_i = 1$ which follows from (4), we obtain (8). Note that all the formulas above are applied for the case $x_i > 0$ only; nevertheless, if $x_i = 0$, we have $U_i'(x_i) x_i = 0$, and therefore, the equation (8) holds for any optimum allocation vector $\vec{x}$.

Finally, we note that in equilibrium, the vector of discrimination weights $\vec{C}$ and the vector of bids $\vec{w}$ are functions of the equilibrium allocation $\vec{x}$ given as follows: for every i,

$$C_i = x_i + \frac{R(\vec{x})}{U_i'(x_i)} \text{ and } w_i = \frac{R(\vec{x})}{U_i'(x_i) x_i + R(\vec{x})} U_i'(x_i) x_i.$$

Equilibrium of Weighted Payment Auction. The analysis follows similar steps as for the weighted bid auction; in this case the revenue at Nash equilibrium can be represented as an explicit function of the allocation vector $\vec{x}$. Given a discrimination weight C_i, user i solves the following surplus maximization problem:

$$\textbf{USER:} \quad \max_{w_i \geq 0} U_i \left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C \right) - C_i w_i. \tag{9}$$

Lemma 2 *Given a vector of discrimination weights $\vec{C}$, there is a unique allocation $\vec{x}$ corresponding to the unique Nash equilibrium. Conversely, given an allocation $\vec{x}$, such that $\sum_i x_i = C$, there is a vector $\vec{C}$ of discrimination weights such that $\vec{x}$ is an outcome. Furthermore, the corresponding revenue $R(\vec{x})$ is given by*

$$R(\vec{x}) = \sum_i U_i'(x_i) x_i \left(1 - \frac{x_i}{C}\right).$$

Proof. For the USER problem (9) we have that either $w_i = 0$ or the derivative of the objective function is zero, i.e.

$$U_i'(x_i)\frac{\sum_j w_j - w_j}{(\sum_j w_j)^2} - C_i = 0.$$

Combining with the allocation rule of weighted bid auction, it is easy to observe that the last equality is equivalent to

$$U_i'(x_i)\frac{C - x_i}{\sum_j w_j} = C_i.$$

Therefore,

$$R = \sum_i C_i w_i = \sum_i U_i'(x_i)(C - x_i)\frac{w_i}{\sum_j w_j}$$

which combined with $x_i = Cw_i/\sum_j w_j$ yields the asserted revenue in the lemma. It is straightforward to see that given $\vec{x}$ such that $\sum_i x_i = C$, we can find C_i and w_i such that the condition of the Nash equilibrium is satisfied at $\vec{x}$. ■

3 Revenue

Revenue of Traditional Proportional Sharing. We demonstrate poor performance of traditional proportional sharing with respect to revenue for the example of a parking-lot network that is a canonical example used in the context of networking (Figure 2). The resource consists of a series of $n \geq 1$ links, each of a capacity $C > 0$ (without loss of generality we assume $C = 1$). The example consists of $n+1$ users; user 0 is a *multi-hop user* that requires a connection through links $1, 2, \ldots, n$ while a user i is a *single-hop user* that requires a connection through link i, for $i = 1, 2, \ldots, n$. User utility functions are assumed to be α-fair, i.e. for $\alpha > 0$ and $\alpha \neq 1$, we have $U_i(x) = \frac{w_i^\alpha}{1-\alpha} x^{1-\alpha}$, and $U_i(x) = w_i \log(x)$, for $\alpha = 1$, where $w_i > 0$ (in our example, we consider a symmetric case where $w_i = 1$, for every user i). The resource constraints in this example are $x_i + x_0 \leq 1$, for every link $i = 1, 2, \ldots, n$.

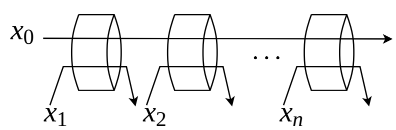

Figure 2: Parking-lot example.

Under proportional sharing mechanism, user 0 submits a bid for each of n links while user i, $i = 1, 2, \ldots, n$, only bids for the link i. Using the known conditions for Nash equilibrium of the game (e.g. [11, 19]), one can show that there is a unique equilibrium that is a solution of the following problem:

$$\begin{aligned} &\text{maximize } \sum_{i=0}^{n} \int_0^{x_i} U_i'(y)(1-y)dy \\ &\text{over } \vec{x} \in \mathbb{R}_+^{n+1} \\ &\text{subject to } x_i + x_0 \leq 1,\ i = 1, 2, \ldots, n. \end{aligned}$$

Furthermore, the revenue at Nash equilibrium allocation $\vec{x}$ is the sum of all the bids on every links. As shown in [19], on link i, the total bid is equal to $U_i'(x_i)(1-x_i)$, thus the revenue of the traditional proportional sharing mechanism is $\sum_{i=1}^{n} U_i'(x_i)(1-x_i)$. For our example, by a straightforward calculation, one can show that Nash equilibrium allocation is x_0 for user 0 and $1 - x_0$ for each user $i = 1, 2, \ldots, n$, where $x_0 = \frac{1}{1+n^{1/(\alpha+1)}}$. The revenue of this mechanism is $\frac{nx_0}{(1-x_0)^\alpha}$. For large network size n, it is not difficult to observe that the revenue is $O(n^{\frac{\alpha}{\alpha+1}})$. Hence, the revenue is $o(n)$ for every fixed $\alpha > 0$. The intuition behind why the revenue is low is quite clear; for large network size n, user 0 has to compete with many users, therefore, the payment on each link that this user can afford is becoming smaller. Consequently, the competition at each link is reduced. If, on the other hand, a fixed price scheme is used, the provider can charge a unit price on each link and receive a revenue of n.

The question that we investigate in this section is how much revenue the weighted bid auction can achieve in comparison with a best fixed price scheme (with discrimination). We will answer this question in the remainder of this section.

Revenue of Weighted Bid Auction. We will compare the revenue of weighted bid auction with that of a benchmark that uses a standard fixed-price scheme [24]. In this pricing scheme, the provider charges user-specific prices per unit of resource for different users. Suppose p_i is the price per unit of resource for user i, then user i surplus maximization problem is $\max U_i(x_i) - p_i x_i$ over $x_i \geq 0$. The solution of this problem is given by $U_i'(x_i) = p_i$. Therefore, the revenue of the provider is $R = \sum_i p_i x_i = \sum_i U_i'(x_i)x_i$, and thus the optimal revenue is $R^* = \max\{\sum_i U_i'(x_i)x_i : \vec{x} \in \mathcal{P}\}$.

However, comparing with such a benchmark would be too ambitious because with auction-based allocation the provider cannot announce fixed prices, but instead the prices are induced from the demand of users. Thus, instead, we will compare our revenue with R^* in a setting where *some users are excluded*. That is, we will compare the revenue achieved by an auction for n users with the revenue achieved by fixed prices for $n-k$ users, for $0 < k < n$. We note that this is a standard way of revenue comparison in the theory of auctions [9].

In the parking-lot example introduced above, if we consider n large enough, then the optimal revenue is of order n, which can be achieved if the provider charges every single-hop user a unit price of 1 and charges the multi-hop user a unit price of n. Now, if we exclude an arbitrary set of $k < n$ users, then the optimal revenue is $n-k$. Therefore, if k is much smaller compared with n, then one can think of this as a *large market* situation where the effect of the fact that a few users do not participate in the market is negligible.

Having discussed the intuition, we can now state our main result on the revenue guarantee of weighted bid auctions. Let R^*_{n-k} be the optimal revenue under our benchmark, i.e.

$$R^*_{n-k} = \min_{S\subset\{1,\ldots,n\}:\ |S|=n-k} \max_{\vec{x}\in\mathcal{P}} \sum_{i\in S} U_i'(x_i)x_i.$$

The revenue guarantee of weighted bid auctions can be stated as follows.

Theorem 1 *Suppose that for each i, $U_i'(x)x$ is a concave function. Let R be the optimum revenue of the weighted bid allocation mechanism, then*

$$\textit{for every } 1 \leq k < n: \ R \geq \frac{k}{k+1} R^*_{n-k}.$$

The revenue guarantee of the theorem above is rather strong. In particular, by taking as a benchmark the system with just one user excluded, we obtain that the revenue under weighted bid auction is a factor $1/2$ of the revenue under standard price discrimination with one less user (whose exclusion reduced the revenue the most). Informally, the result tells us that for systems with many users with comparable utility functions, the revenue under weighted bid auction would be close to the revenue under standard price discrimination. As discussed above such a guarantee cannot be provided by traditional proportional sharing.

Proof. For $R(\vec{x})$ given by (2), it is easy to observe that for every $\vec{x} \in \mathcal{P}$,

$$\sum_i U_i'(x_i)x_i - \max_j U_j'(x_j)x_j \leq R(\vec{x}) < \sum_i U_i'(x_i)x_i. \tag{10}$$

Suppose that for each $1 \leq k < n$, there exists $\vec{x} \in \mathcal{P}$ such that both of the following two conditions hold

(i) $\sum_i U_i'(x_i)x_i \geq R^*_{n-k}$;

(ii) $U_1'(x_1)x_1 = \cdots = U_{k+1}'(x_{k+1})x_{k+1} \geq \cdots \geq U_n'(x_n)x_n$,

where, without loss of generality, the users are enumerated such that $U_1'(x_1)x_1 \geq \cdots \geq U_n'(x_n)x_n$. Under conditions (i) and (ii), the theorem follows because

$$\begin{aligned} R(\vec{x}) &\geq \sum_i U_i'(x_i)x_i - \max_j U_j'(x_j)x_j \\ &\geq \frac{k}{k+1}\sum_i U_i'(x_i)x_i \geq \frac{k}{k+1} R^*_{n-k}. \end{aligned}$$

We show that such a vector $\vec{x}$ exists by induction over k. *Base step*: $k = 0$. The vector $\vec{x}$ that maximizes $\sum_i U_i'(x_i)x_i$ over $\vec{x} \in \mathcal{P}$ satisfies both conditions. *Induction step*: Let $\vec{x} \in \mathcal{P}$ be a vector such that both condition (i) and condition (ii) hold for k. We then show that there exists another vector in $\mathcal{P}$ such that these conditions hold for $k+1$. Note that $R^*_{n-k} \geq R^*_{n-(k+1)}$ as allowing to exclude a larger set of users cannot increase R^*_{n-k}. In the following, without loss of generality, we assume that users are enumerated such that $U_1'(x_1)x_1 \geq \cdots \geq U_n'(x_n)x_n$. Let $\vec{y} \in \mathcal{P}$ be an optimum solution of the provider's problem under our benchmark pricing scheme and the constraint $y_1 = \ldots = y_{k+1} = 0$, i.e. with users $1, 2, \ldots, k+1$ excluded. We have that $\sum_i U_i'(y_i)y_i \geq R^*_{n-(k+1)}$.

Now, let us consider the vector $\vec{v}(t)$ defined by

$$\begin{aligned}\vec{v}(t) = &(1-t)\cdot(U_1'(x_1)x_1, \ldots, U_n'(x_n)x_n) + \\ &+ t\cdot(U_1'(y_1)y_1, \ldots, U_n'(y_n)y_n), \quad \text{for } t \in [0,1].\end{aligned}$$

Note that as t increases from 0, the $k+1$ largest coordinates of $\vec{v}(t)$ decrease, while all the other coordinates either increase or do not change. Thus, there exists $t^* \in [0,1]$ such that the largest $k+2$ coordinates of $\vec{v}(t)$ are equal. Furthermore, as $\sum_i U_i'(x_i)x_i \geq R^*_{n-(k+1)}$ and $\sum_i U_i'(y_i)y_i \geq R^*_{n-(k+1)}$, we have that $\sum_i v_i(t^*) \geq R^*_{n-(k+1)}$. Finally, since for each i, $U_i'(x_i)x_i$ is concave, there exists a vector $\vec{z} \in \mathcal{P}$ such that $(U_1'(z_1)z_1, \ldots, U_n'(z_n)z_n) = \vec{v}(t^*)$. By this, we showed that the vector $\vec{z}$ satisfies conditions (i) and (ii) for $k+1$ which completes the proof. ∎

Remark For the weighted payment auction, observe that the maximum revenue is given by

$$\max\left\{\sum_i U_i'(x_i)x_i(1 - x_i/C) : \vec{x} \in \mathbb{R}^n_+,\ \sum_i x_i = C\right\}.$$

On the other hand, the optimal revenue under the price discrimination scheme is

$$R^* = \max\left\{\sum_i U_i'(x_i)x_i : \vec{x} \in \mathbb{R}^n_+,\ \sum_i x_i = C\right\}.$$

Therefore, given $k \geq 1$, there will be at most k users who get at least C/k. For other users, we have $x_i \leq C/(k+1)$, thus $U_i'(x_i)x_i(1 - x_i/C) \geq \frac{k}{k+1}U_i'(x_i)x_i$. If we choose C_i such that the outcome of the game is the allocation vector $\vec{x}$ that maximizes R^*, we can also obtain a similar revenue lower bound for the case of weighted pay auction with an additional constraint that $U_i'(x_i)x_i$ is nondecreasing.

4 Efficiency for Linear User Utility Functions

In this section, we analyze the efficiency of the system for the case of single provider and linear user utility functions. From a technical point of view, although this result is for a special class of utilities, it provides us with basic techniques for a more general result established in the next section.

4.1 Efficiency of Weighted Bid Auction

We show the following theorem.

Theorem 2 *Assume that the provider maximizes the revenue and for each user i, the utility function is linear, $U_i(x) = v_i x$, for some $v_i > 0$. Then, the worst-case efficiency is $1/(1+2/\sqrt{3})$ (approx. 46%). Furthermore, this bound is tight.*

Remark Before proving the theorem, note that the worst-case efficiency can be achieved asymptotically as the number of users n tends to infinity. One example is when we have the resource constraint $\sum_i x_i \leq 1$, and there is a unique user with largest marginal utility, say this is user 1, and all other users have identical marginal utilities equal to $(2-\sqrt{3})^2 v_1 \approx 0.0718 v_1$. At the Nash equilibrium, user 1 obtains 42.26% of the resource and the rest is equally shared by the remaining users. Thus, the efficiency loss occurs only when there is an unbalance in the marginal utilities by the users. One can actually show that when there is a higher competitiveness among the users, the efficiency increases. More precisely, we show that if there are at least k users with the largest marginal utility, then the efficiency is at least $1-\frac{1}{2k}+o(1/k)$. The proof of this result is Appendix C.

We first need the following lemma about the *quasi-concavity* of the objective function optimized by the provider. Recall that $R(\vec{x})$ is the function given by (2). Let R^* be the optimum revenue, i.e. $R^*=\max\{R(\vec{x}):\ \vec{x}\in\mathcal{P}\}$. We note the following fact.

Lemma 3 *The set $\mathcal{L}_\mu := \{\vec{x}\in\mathbb{R}^n_+ : R(\vec{x})\geq\mu\}$ is convex, for every $\mu\in[0,R^*]$.*

Proof. We want to show that

$$\mathcal{L}_\mu := \{\vec{x}\in\mathbb{R}^n_+ : R(\vec{x})\geq\mu\}$$

is convex, where

$$\sum_i \frac{v_i x_i}{v_i x_i + R(\vec{x})} = 1.$$

It is clear that $R(\vec{x})$ is a monotone increasing function in each x_i, therefore if $\vec{y}\geq\vec{x}$, and $\vec{x}\in\mathcal{L}_\mu$, then also $\vec{y}\in\mathcal{L}_\mu$.

It is enough to see that given $\vec{x}$ and $\vec{y}$ such that $R(\vec{x})=R(\vec{y})=\mu$ then for every other vector $\vec{z}$ on the interval connecting $\vec{x}$ and $\vec{y}$ we have $R(\vec{z})>\mu$. Since $R(\vec{z})$ is a monotone function in each z_i, it is enough to prove that

$$\sum_i \frac{v_i z_i}{v_i z_i+\mu}\geq 1.$$

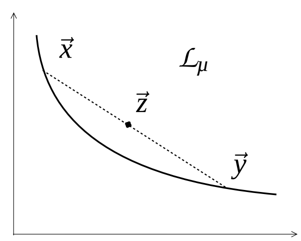

Figure 3: Convexity of the revenue.

Assume $\vec{z}=\alpha\vec{x}+(1-\alpha)\vec{y}$. Since the function $\frac{v_i z_i}{v_i z_i+\mu}$ is concave for every i, we have

$$\frac{v_i z_i}{v_i z_i+\mu}\geq\alpha\frac{v_i x_i}{v_i x_i+\mu}+(1-\alpha)\frac{v_i y_i}{v_i y_i+\mu}.$$

Summing over i, we obtain the desired inequality.

■

In the remainder of this section, we prove Theorem 2.

Proof of Theorem 2 The example showing that the bound is tight is given in the remark above; we now prove that the efficiency is at least $1/(1+2/\sqrt{3})$.

Since for every $\vec{x}\in\mathcal{P}$, $R(\vec{x})\leq R^*$, the two convex sets $\mathcal{L}_{R^*}$ and $\mathcal{P}$ do not have common interior points. Let H be a hyperplane that weakly separates these two sets. This hyperplane can be written as

$$\sum_i \gamma_i x_i = 1,\ \text{with } \gamma_i\geq 0 \text{ for each } i. \tag{11}$$

Consider the game where the provider has the feasible set $\mathcal{Q}=\{\vec{x}\in\mathbb{R}^n_+ : \sum_i\gamma_i x_i\leq 1\}$, then the allocation that maximizes the revenue over $\mathcal{Q}$ is the same as in the original game. Since $\mathcal{P}\subset\mathcal{Q}$, the optimal social welfare of the

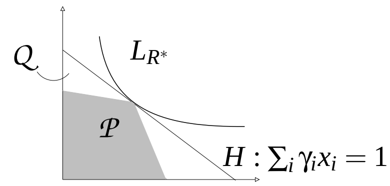

Figure 4: Reduction to a simple constraint.

new game is at least the social welfare of the original game. Therefore, it is enough to prove a lower bound on the efficiency for the class of games where the provider has the feasible set Q. See Figure 4.

The observation above allows us to reduce the analysis to simpler optimization problems. In particular, the optimal social welfare in this new game is $\max_i v_i/\gamma_i$; the condition for Nash equilibrium, as argued above, is the condition for $\vec{x}$ to maximize $R(\vec{x})$ over $\vec{x} \in \mathbb{R}^n_+$ such that $\sum_i \gamma_i x_i = 1$, which we derive in the following. Taking the partial derivative with respect to x_j on both sides in (2), with $U_i(x_i) = v_i x_i$, we have

$$\frac{\partial}{\partial x_j} \sum_i \frac{v_i x_i}{v_i x_i + R} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial}{\partial x_j} \frac{v_j x_j}{(v_j x_j + R)} - \sum_{i \neq j} \frac{\partial R}{\partial x_j} \frac{v_j x_i}{(v_i x_i + R)^2} = 0.$$

Note that

$$\frac{\partial}{\partial x_j} \frac{v_j x_j}{v_j x_j + R} = \frac{R v_j}{(v_j x_j + R)^2} - \frac{\partial R}{\partial x_j} \frac{v_j x_j}{(v_j x_j + R)^2}.$$

Thus, we have

$$\frac{R v_j}{(v_j x_j + R)^2} = \frac{\partial R}{\partial x_j} \cdot \sum_i \frac{v_i x_i}{(v_i x_i + R)^2}.$$

Since $R(\vec{x})$ achieves the optimum value R^* over the set $\{\vec{x} \in \mathbb{R}^n_+ : \sum_i \gamma_i x_i \leq 1\}$, we have either $x_j = 0$ or $\frac{\partial}{\partial x_j} R = \lambda \gamma_j$ where $\lambda \geq 0$ is a parameter (the Lagrange multiplier associated to the constraint $\sum_i \gamma_i x_i \leq 1$). It follows that

$$\begin{aligned} &\text{either } x_i = 0 \\ &\text{or } \frac{v_i/\gamma_i}{(v_i x_i + R^*)^2} = \frac{\lambda}{R^*} \sum_i \frac{v_i x_i}{(v_i x_i + R^*)^2} = p. \end{aligned} \tag{12}$$

By this, we obtain a condition that at the Nash equilibrium allocation $\frac{v_i/\gamma_i}{(v_i x_i + R^*)^2}$ are equal to a common value $p > 0$. Therefore, if v_i/γ_i is large then the denominator $(v_i x_i + R^*)^2$ needs to be large as well. At the same time, the optimal solution of social welfare distributes all the resource to the user with the highest v_i/γ_i. This is the intuition for the fact that the efficiency is bounded by a constant.

First we will scale the variables to make the equations easier to follow. We will use a new set of variables, namely $z_i = \gamma_i x_i$ and $a_i = v_i/\gamma_i$. One way to think about this new variables is to think of another game where the resource constraint is $\sum_i z_i = 1$ and user i's utility is $a_i z_i$. Without loss of generality, we assume that $a_1 = \max_i a_i$. The optimal social welfare is

$$W_{\mathrm{OPT}} = \max_{\vec{x}: \sum_i \gamma_i x_i = 1} \sum_i v_i x_i = \max_{\vec{z}: \sum_i z_i = 1} \sum_i a_i z_i = a_1.$$

We now introduce new variables y_i, $i = 1, 2, \ldots, n$, defined by

$$y_i = v_i x_i / (v_i x_i + R^*) = a_i z_i / (a_i z_i + R^*).$$

Because of (2), we have $\sum_i y_i = 1$. The goal of introducing these variables is to bound the optimal social welfare and the social welfare of a Nash equilibrium as functions of y_i. Now, from $y_i = a_i z_i / (a_i z_i + R^*)$, we have

$$a_i z_i = R^* \frac{y_i}{1 - y_i} \text{ and } z_i = R^* \frac{y_i}{a_i(1 - y_i)}.$$

Next, we are going to bound the social welfare of a Nash equilibrium and the optimal solution.

The social welfare at the Nash equilibrium, which we denote as W_{NASH}, can be bounded using the relations above as follows

$$\begin{aligned} W_{\mathrm{NASH}} &= \sum_i a_i z_i = R^* \sum_i \frac{y_i}{1-y_i} \\ &\geq R^* \left(\frac{y_1}{1-y_1} + \sum_{i\geq 2} y_i \right) = R^* \left(\frac{y_1}{1-y_1} + 1 - y_1 \right) \\ &\geq R^* \frac{y_1^2 - y_1 + 1}{1-y_1}. \end{aligned} \tag{13}$$

The above inequality uses the fact that $\frac{y_i}{1-y_i} \geq y_i$ and $\sum_i y_i = 1$.

On the other hand, the optimal social welfare, as argued above, is

$$W_{\mathrm{OPT}} = \max_i a_i = a_1.$$

To bound a_1 as a function of y_i, we multiply a_1 with $\sum_i z_i$, which is 1, and use the relation between z_i and y_i to have W_{OPT} as a function of y_i. Specifically,

$$W_{\mathrm{OPT}} = a_1 = a_1 \left(\sum_i z_i \right) = a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)}. \tag{14}$$

Now, we use the condition for Nash equilibrium. (Note that this is the only place in the proof that uses (12).) First we rewrite the condition for the variables z_i and a_i. Replacing $a_i = v_i/\gamma_i$ and $v_i x_i = a_i z_i = R^* \frac{y_i}{1-y_i}$ in to the condition for Nash equilibrium (12), we can derive

$$\text{either } y_i = 0 \text{ or } \frac{a_i(1-y_i)^2}{R^{*2}} = p > 0.$$

From this condition, we have $a_i(1-y_i)^2 = a_1(1-y_1)^2$ whenever $y_1, y_i > 0$, hence $a_i(1-y_i) = \frac{a_1(1-y_1)^2}{1-y_i}$. Replacing this equality in the optimal social welfare (14), we have

$$\begin{aligned} W_{\mathrm{OPT}} = a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)} &= \frac{R^*}{(1-y_1)^2} \sum_i y_i(1-y_i) \\ &\leq \frac{R^*}{(1-y_1)^2} \left(y_1(1-y_1) + \sum_{i\geq 2} y_i \right). \end{aligned}$$

The last inequality uses the fact that $y_i(1-y_i) \leq y_i$. Using this and replacing $\sum_{i\geq 2} y_i = 1 - y_1$, we obtain

$$W_{\mathrm{OPT}} \leq \frac{R^*}{(1-y_1)^2} (y_1(1-y_1) + 1 - y_1) = R^* \frac{1-y_1^2}{(1-y_1)^2}. \tag{15}$$

From (13) and (15), we have the following lower bound for the efficiency

$$\frac{W_{\mathrm{NASH}}}{W_{\mathrm{OPT}}} \geq \frac{y_1^2 - y_1 + 1}{y_1 + 1}.$$

By a simple calculus, one can show that the right-hand side is at least $1/(1+2/\sqrt{3})$, which is what we needed to prove. ∎

4.2 Efficiency of Weighted Payment Auction

For weighted payment auction, we establish the following result.

Theorem 3 *Assuming that the provider maximizes the revenue, the efficiency is bounded by 1/2 for weighted payment auction, and this bound is tight.*

Proof. The revenue $R(\vec{x})$ at the Nash equilibrium allocation $\vec{x}$ of the weighted payment auction is given by Lemma 2. Therefore, $R(x)$ is maximized over $\vec{x} \in \mathbb{R}_+^n$ subject to the constraint $\sum_i x_i = C$, when

$$\frac{d}{dx_i}\left(U_i'(x_i)x_i(1-\frac{x_i}{C})\right) = \lambda > 0 \text{ whenever } x_i > 0.$$

In the case of linear utility functions, $U_i(x_i) = v_i x_i$, this means

$$v_i\left(1-2\frac{x_i}{C}\right) = \lambda > 0 \text{ whenever } x_i > 0.$$

We assume that $v_1 \geq v_2 \geq \cdots \geq v_n$. From the condition above, one has:

$$\text{if } x_i > 0, \text{ then } v_i = \frac{\lambda}{1-2\frac{x_i}{C}} > \lambda = v_1\left(1-2\frac{x_1}{C}\right).$$

Therefore, the social welfare at Nash equilibrium allocation $\vec{x}$ is $\sum_i v_i x_i$, and is at least

$$v_1 x_1 + v_1\left(1-2\frac{x_1}{C}\right)\left(\sum_{i=2}^n x_i\right) = v_1 x_1 + v_1\left(1-2\frac{x_1}{C}\right)(C-x_1).$$

By a straightforward calculation, one can see that compared with the optimal social welfare, which is Cv_1, the value above is at least 1/2 of the optimal social welfare.

The bound can be achieved, for example, for $v_1 = 1$ and $v_2 = v_3 = \cdots = v_n = \frac{1}{n}$ as n goes to infinity.

■

5 Extension to Multiple Providers and General User Utilities

In this section, we will extend the efficiency result of the previous section to the case of multiple competing providers and a broad class of utility functions. Our main result in this section shows a surprising fact that even in complex competitive environments, the efficiency can be bounded by a constant independent of the number of providers and users (Theorem 4). We first define the framework for multiple providers.

5.1 Multiple Providers

In a system of multiple competing providers, each provider allocates resources according to the weighted proportional allocation. We assume that each provider k has constraints specified by a convex set $\mathcal{P}_k$ that is allowed to be different from one provider to another. We assume that each user can receive resources from any provider and is concerned only with the total allocation received over all providers. Note that both of these assumptions can be relaxed, as we can encode some constraints in the convex set $\mathcal{P}_k$. We will use the following notation. Let x_i^k denote the allocation to user i by provider k. For each user i, the utility of an allocation $(x_i^k,\ k=1,\ldots,m)$ is $U_i(\sum_k x_i^k)$. Let $x_i = \sum_k x_i^k$ denote the total allocation to user i over all providers. We denote with $x_i^{-k} = x_i - x_i^k$ the total allocation to user i over all providers except provider k. See Fig. 5 for an illustration.

Let $\vec{x} = (x_i^k,\ i=1,\ldots,n,\ k=1,\ldots,m)$ be an allocation under weighted proportional sharing mechanism. It is analog to the argument in Section 2 that given $\vec{x}$, each provider k can find the weights $(C_1^k, C_2^k, \ldots, C_n^k)$ such that $\vec{x}$ is the equilibrium of the weighted proportional sharing in the multiple providers' setting. In this setting the payment

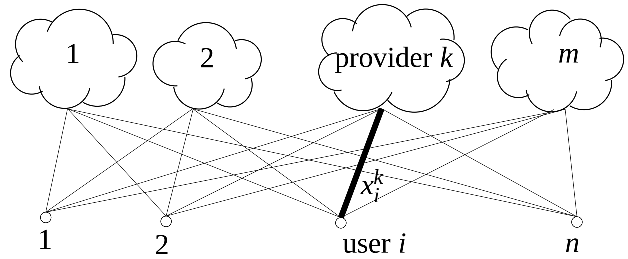

Figure 5: The setting of multiple providers competing in offering resources to users.

of user i to provider k is w_i^k, and the user's goal is to maximize $U_i(\sum_k x_i^k) - \sum_k w_i^k$, where $x_i^k = C_i^k w_i^k / \sum_i w_i^k$. The provider k, on the other hand, obtains the revenue R^k, which satisfies the following

$$\sum_{i=1}^{n} \frac{U_i'(x_i^{-k} + x_i^k) x_i^k}{U_i'(x_i^{-k} + x_i^k) x_i^k + R^k} = 1. \tag{16}$$

In order to gain some intuition, note that if for every i, $U_i(x)$ is a concave function, then $U_i'(x_i^{-k} + x_i^k)$ decreases with x_i^{-k}. From this, we can see that the marginal utility for a user with a provider k decreases if the user already received allocations from other providers. As a result, provider k may extract smaller revenue due to competition with other providers. With this in mind, we now define an equilibrium in the case of multiple providers.

Definition 1 *We call $\vec{x}$ an equilibrium allocation if for every k, the allocation vector $\vec{x}^k = (x_1^k, \ldots, x_n^k)$ maximizes R^k given by (16) over the set $\mathcal{P}_k$.*

We note that in the multiple providers setting, we can think of the game as the provider k's strategy set is $\mathcal{P}_k$. The discrimination weights and the revenue then can be calculated according to the allocation vector $\vec{x}$ of all providers. With these discrimination weights, under users' selfish behavior $\vec{x}$ will be an outcome of the game. From the providers' perspective, an equilibrium allocation is an allocation $\vec{x}$ where no provider has an incentive to unilaterally change its allocation vector. Note that when there is only one provider, this game is the same as the two-stage Stackelberg game considered in Section 2.

5.2 A Class of Utility Functions

We introduce a class of utility functions defined as follows.

Definition 2 *Let $U(x)$ be a non-negative, increasing, and concave utility function and let $x_0 \geq 0$ be the value maximizing $U'(x)x$. We call $U(x)$, δ-utility, if, in addition, the following two conditions hold: (i) $U'(x)x$ is a concave function over $[0, x_0]$, and (ii) there exists $\delta \in [0, \infty)$, such that, for every $a \in [0, x_0]$,*

$$U(b) - [U'(a)a]'b \leq \delta U(a)$$

where b is such that $U'(b) = [U'(a)a]' = U'(a) + U''(a)a \geq 0$.

In Figure 6, we show geometric interpretations of the latter definition.

While the definition may appear somewhat technical, in fact, it has strong connections with the theory of third-degree price discrimination [24], which we discuss in Appendix D. Furthermore, the class accommodates many utility functions commonly considered in literature. Specifically, we can show that a linear function or a truncated linear function is 0-utility, a polynomial $U(x) = (c + x)^\alpha$ for $c \geq 0$ is a $\frac{e}{2}$-utility for any $0 \leq \alpha \leq 1$, or a logarithmic function is a 2-utility; we provide a detailed list and proofs in Appendix E. We briefly comment that truncated linear utility functions or logarithmic functions were considered representative of real-time traffic requirements in communication networks [23], concave marginal utilities were considered in [20], polynomial utility functions were used in many models of economics [24], α-fair utility functions [14] were widely used in the context of communication network resource sharing and a class of utility functions that characterize TCP-like connections [12]. Finally, we note the following result whose proof is available Appendix B.1.

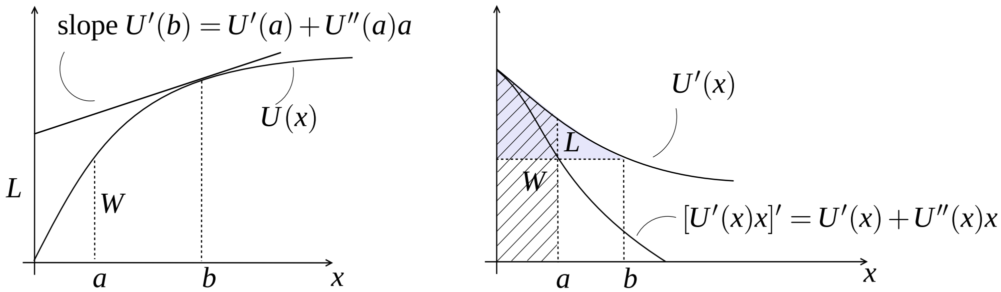

Figure 6: $\frac{L}{W} \leq \delta$ where (left) L and W are lengths of the line segments and (right) L is the shaded and W is the hatched area.

Lemma 4 *If f and g are δ-utilities, then so are: $c \cdot f$, for $c > 0$; truncation of f with a positive number; and $f + g$.*

As a consequence, every polynomial of the form $\sum_i a_i x^{\alpha_i}$ where $a_i > 0$ and $0 \leq \alpha_i \leq 1$ is a $\frac{e}{2}$-utility function.

5.3 Efficiency Bound

We now state and prove our main theorem on the efficiency of weighted bid auctions.

Theorem 4 *Assume that for every user i and every $a \geq 0$, $U_i'(x+a)x$ is a continuous and concave function. Then, there exists an equilibrium in the case of multiple providers defined as above. Furthermore if $U_i(a+x)$ are δ-utilities, then the efficiency at any equilibrium is at least $1/(1 + 2/\sqrt{3} + \delta)$.*

Note that when the utility functions are linear, i.e. $\delta = 0$, we have Theorem 2 as a special case. The result of Theorem 4 is rather surprising as it is not a priori clear that in a complex system where both users and providers aim at selfishly maximizing their individual payoffs (objectives which often conflict each other), the efficiency would be bounded by a constant that is independent of the number of users and the number of providers.

Before going into the proof of Theorem 4 provided below, we briefly describe the main ideas. The key idea of the proof is to bound the social welfare of the system, which is a complicated optimization problem over the Minkowski sum of the sets $\mathcal{P}_k$, where the resource set $\mathcal{P}_k$ of the provider k can be different. If the utility functions are linear, then this optimization problem can be *separated* into a collection of optimization problems over each set $\mathcal{P}_k$, and the optimal value is the sum of these optimal values. Using this idea, we will bound the utility function by an affine function that is a tangent to the concave utility at the point defined by the value at the Nash equilibrium. This idea is illustrated in Figure 7. It will be shown that because of the property of δ-utilities, the value a_i in the figure is at most $\delta U_i(x_i)$, which will be a key inequality of the proof.[2]

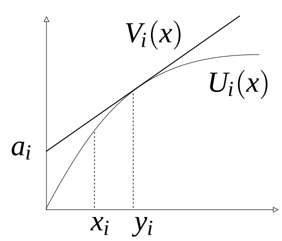

Figure 7: The key bounding of $U_i(x)$ with the affine function $V_i(x)$ such that $V_i'(y_i) = U_i'(x_i) + U_i''(x_i)\max_k x_i^k$ and $V_i(y_i) = U_i(y_i)$.

[2]We note that our proof technique is rather general and that a similar result for weighted payment auctions can also be obtained using this framework. However, we omit to provide details as our focus is on weighted bid auctions that allow for much more general resource constraints.

Proof. The proof for the first part of the theorem about the existence of a Nash equilibrium uses standard fixed point theorem argument. A formal proof is given in Appendix refapp:fix.

For the second part, the key idea of the proof is to bound the social welfare by an affine function which allows separating the maximization over $(\vec{x}^1,\dots,\vec{x}^m)\in\sum_k \mathcal{P}_k$ to maximizations over the sets $\mathcal{P}_k$, where $\sum_k \mathcal{P}_k := \{\vec{z}^1+\cdots+\vec{z}^m : \vec{z}^k\in\mathcal{P}_k, k=1,\dots,m\}$ is the Minkowski sum of the sets $\mathcal{P}_k$. Once the optimization problem is separated, we can use a similar bound as in Section 4 (see Lemma 5 below) as a subroutine to prove the theorem. Now, let

$$v_i^k = U_i'(x_i) + U_i''(x_i)x_i^k, \text{ for each } i, \text{ and } v_i = \min_k v_i^k.$$

Since for every i, $U_i(x)$ is a concave function, $U_i''(x)$ is non-positive, and thus,

$$v_i = U_i'(x_i) + U_i''(x_i)(\max_k x_i^k) \ge U_i'(x_i) + U_i''(x_i)x_i.$$

The last inequality is because of the fact $x_i = \sum_k x_i^k$.

Now, let us define $V_i(x) = a_i + v_i x$ where a_i is chosen so that $V_i(x)$ is a tangent to $U_i(x)$. Let y_i be the point at which the functions $V_i(x)$ and $U_i(x)$ intersect. We will use $V_i(x)$ as an upper bound of $U_i(x)$ for every $x\ge 0$. We have $a_i = U_i(y_i) - (U_i'(x_i)+U_i''(x_i)x_i)y_i$. By the definition of δ-utility functions, $a_i \le \delta U_i(x_i)$.

Therefore,

$$\sum_i a_i \le \delta \sum_i U_i(x_i). \tag{17}$$

Since $U_i(x)$ is a non-negative concave function, we have $U_i(x)\le V_i(x)$. Hence,

$$\begin{aligned}\max_{\vec{z}\in\sum_k \mathcal{P}_k} \sum_i U_i(z_i) &\le \max_{\vec{z}\in\sum_k \mathcal{P}_k} \sum_i V_i(z_i)\\ &= \sum_i a_i + \max_{\vec{z}\in\sum_k \mathcal{P}_k} \sum_i v_i z_i\\ &= \sum_i a_i + \sum_k \max_{\vec{z}\in\mathcal{P}_k} \sum_i v_i z_i. \end{aligned}\tag{18}$$

The last is the key inequality as it enables us to use the fact that $v_i z_i$ are linear functions, therefore, instead of considering the maximization over the set $\sum_k \mathcal{P}_k$ we can bound $\sum_i v_i z_i$ over each $\mathcal{P}_k$.

By similar arguments as in the proof of Theorem 2, we can prove the following lemma whose proof is provided in Appendix B.3.

Lemma 5 *For ever k,*

$$\sum_i U_i'(x_i)x_i^k \ge \frac{1}{1+2/\sqrt{3}} \max_{z\in\mathcal{P}_k} \sum_i v_i^k z_i. \tag{19}$$

We now use this lemma to prove our main result. On the one hand, if we sum the left-hand side of (19) over all k, we have

$$\sum_{k,i} U_i'(x_i)x_i^k = \sum_i U_i'(x_i)x_i \le \sum_i U_i(x_i) \tag{20}$$

where the last inequality is true because $U_i(x)$ is a non-negative and concave function for every i. On the other hand, if we sum the right-hand side of (19) over all k, we obtain

$$\sum_k \frac{1}{1+2/\sqrt{3}} \max_{z\in\mathcal{P}_k} \sum_i v_i^k z_i \ge \frac{1}{1+2/\sqrt{3}} \sum_k \max_{z\in\mathcal{P}_k} \sum_i v_i z_i \tag{21}$$

where in the last inequality, v_i^k are replaced by v_i, which recall is equal to $\min_k v_i^k$.

Combining (19)–(21), we derive

$$\sum_i U_i(x_i) \ge \frac{1}{1+2/\sqrt{3}} \sum_k \max_{z\in\mathcal{P}_k} \sum_i v_i z_i$$

$$\Rightarrow (1+2/\sqrt{3})\sum_i U_i(x_i) \geq \sum_k \max_{z\in\mathcal{P}_k} \sum_i v_i z_i. \qquad (22)$$

Finally, from (17), (18) and (22), we have

$$\max_{\vec{z}\in\sum_k \mathcal{P}_k} \sum_i U_i(z_i) \leq (\delta+1+2/\sqrt{3})\sum_i U_i(x_i)$$

which establishes the asserted result. ■

6 Convergence and Distributed Algorithms

In this section we demonstrate how a provider may adjust user discrimination weights by an iterative algorithm whose limit points are Nash equilibrium points of the resource competition game that we studied in earlier sections. We focus on weighted bid auctions but note that similar type of analysis can be carried out for weighted payment auctions. Our aim in this section is to show how such iterative algorithms can be designed in principle.[3]

One of the main ideas is to relax polyhedron constraints by introducing a *penalty* function $P(\vec{x})$ that is chosen so that to confine the allocation vector $\vec{x}$ within the feasible set specified by our polyhedron constraints $A\vec{x} \leq \vec{b}$. Intuitively, function $P(\vec{x})$ would be chosen to assume small values for every feasible allocation vector $\vec{x}$ that is sufficiently away from the boundary of the feasible set and would grow large as the allocation vector $\vec{x}$ approaches the boundary of the feasible set. We assume that $P(\vec{x})$ is continuously differentiable and convex function. Specifically, we assume that for a collection of functions P_l, one for each of the constraints, we have

$$P(\vec{x}) = \sum_l P_l\left(\sum_j a_{l,j}x_j\right).$$

Indeed, if P_l is continuously differentiable and convex function for every l, then so is P. We define $V(\vec{x}) = R(\vec{x}) - P(\vec{x})$ where $R(\vec{x})$ is the revenue given by (2). The provider's problem is redefined to

$$\textbf{PROVIDER}': \text{maximize } V(\vec{x}) \text{ over } \vec{x}\in \mathbb{R}_+^n.$$

In the remainder of this section, we first show how the provider would adjust discrimination weights assuming that the provider knows user utility functions and establish convergence to the Nash equilibrium points in this case. This provides a baseline dynamics that we then approximate as follows. We consider a provider who a priori does not know user utility functions but estimates the needed information in an online fashion while adjusting the discrimination weights. The main idea here is to use an argument based on *separation of timescales* where discrimination weights are adjusted by the provider at a slower timescale in comparison with the rate at which bids are received from users, allowing the provider to estimate the needed information about the user utility functions for every given set of discrimination weights. We will formulate iterative algorithms as dynamical systems in continuous time as this is standard in previous work, e.g. [13], and it readily suggests practical distributed algorithms.

User Utility Functions a Priori Known. Suppose that user utility functions are a priori known by the provider (this may be the case if profiles of users are known to the provider, e.g. from the history of previous interactions). The provider announces discrimination weights $\vec{C}(t)$ at every time $t \geq 0$ that are adjusted as follows. The provider computes the allocation vector $\vec{x}(t)$ according to the following system of ordinary differential equations, for some $\alpha > 0$,

$$\frac{d}{dt}x_i(t) = \alpha x_i(t)\frac{\partial}{\partial x_i}V(\vec{x}(t)),\ i = 1,2,\ldots,n. \qquad (23)$$

For every time $t \geq 0$, the provider announces to users the discrimination weights $\vec{C}(t)$ where the discrimination weight for user i is:

$$C_i(t) = x_i(t) + \frac{R(\vec{x}(t))}{U_i'(x_i(t))}.$$

[3] Note that it is beyond the scope of this paper to fully specify some of the implementation details such as online estimation of the elasticity of user utility functions and address the stability in presence of feedback delays.

Notice that the right-hand side in the system (23) requires knowledge of the gradient of the revenue function $R(\vec{x})$, which is given by

$$\frac{\partial}{\partial x_i}R(\vec{x}) = \phi(\vec{x})\frac{[U_i'(x_i)x_i]'}{(U_i'(x_i)x_i + R(\vec{x}))^2} \tag{24}$$

where

$$\phi(\vec{x}) = \frac{R(\vec{x})}{\sum_j \frac{U_j'(x_j)x_j}{(U_j'(x_j)x_j+R(\vec{x}))^2}}.$$

The convergence to optimal solution of PROVIDER' is showed in the following result.

Theorem 5 *Suppose that for every user i, $U_i'(x)x$ is continuously differentiable and concave function and that $P(\vec{x})$ is a strictly convex function. Then, every trajectory $(\vec{x}(t),\ t \geq 0)$ of the system (23) converges to a unique maximizer of the function $V(\vec{x})$.*

The proof is based on standard application of Lyapunov stability theorem and is thus omitted here. It amounts to showing that the function V is a Lyapunov function for the system (23) that increases along every trajectory $\vec{x}(t)$, thus implying convergence to the unique maximizer of V.

User Utility Functions a Priori Unknown. We now discuss how the user discrimination weights would be adjusted by a provider who does not a priori know the user utility functions. The key idea is to use a *separation of timescales*: the provider adjusts the discrimination weights at a slower timescale than the timescale at which bids are adjusted by users. Informally speaking, this allows the provider to act as if for every fixed set of discrimination weights, users adjust their bids instantly to Nash equilibrium bids.

From (6) and the allocation rule $x_i = C_i w_i / \sum_j w_j$ one readily observes that the following identities hold at Nash equilibrium:

$$U_i'(x_i)x_i = \frac{Rw_i}{R-w_i} \text{ and } U_i'(x_i)x_i + R = \frac{R^2}{R-w_i}$$

where $R = \sum_j w_j$. Using these identities, we observe that the gradient in (24) can be expressed as follows

$$\frac{\partial}{\partial x_i}R = R\frac{\frac{w_i}{R}\left(1-\frac{w_i}{R}\right)\frac{1}{x_i} + \left(1-\frac{w_i}{R}\right)^2\frac{U_i''(x_i)x_i}{R}}{\sum_j \frac{w_j}{R}\left(1-\frac{w_j}{R}\right)}. \tag{25}$$

Notice that the gradient is fully expressed as a function of the vector of user bids $\vec{w}$ except for the term that involves the second derivative of the user utility function. For the case of linear user utility functions, we have $U_i''(x_i) = 0$ for every i, and thus, in this case the gradient of the revenue $R(\vec{x})$ is fully described by the vector of bids $\vec{w}$.

In general, we assume that at every time $t \geq 0$, the provider sets the user discrimination weight for user i as follows

$$C_i(t) = \frac{R(\vec{w}(t))}{w_i(t)}x_i(t).$$

For the case of linear utility function, $\vec{x}(t)$ is assumed to evolve according to the following system of ordinary differential equations, for some $\alpha > 0$, and every $i = 1, 2, \ldots, n$,

$$\frac{d}{dt}x_i(t) = \alpha\left[v_i(w_i(t), R(\vec{w}(t))) - x_i(t)p_i(\vec{x}(t))\right] \tag{26}$$

where

$$\begin{aligned} v_i(w_i, R) &= R\frac{\frac{w_i}{R}\left(1-\frac{w_i}{R}\right)}{\sum_j \frac{w_j}{R}\left(1-\frac{w_j}{R}\right)} \\ p_i(\vec{x}) &= \sum_l a_{l,i}P_l'\left(\sum_j a_{l,j}x_j(t)\right). \end{aligned}$$

In general, each user i is assumed to adjusts its bid $w_i(t)$ using the following natural dynamics for solving his USER problem:

$$\frac{d}{dt}w_i(t) = U_i'(x_i(t))x_i(t) - R(\vec{w}(t))\frac{\frac{w_i(t)}{R(\vec{w}(t))}}{1-\frac{w_i(t)}{R(\vec{w}(t))}}. \tag{27}$$

The following result establishes convergence for the case of linear user utility functions, which are a priori unknown by the provider.

Theorem 6 *Suppose that user utility functions are linear. For every sufficiently small* $\alpha > 0$, *the allocation vector under system (26)-(27) approximates that of the system (23) with an approximation error diminishing with* α.

We note that this proof is based on applying the *averaging theory* of non-linear dynamical systems [16]. It establishes global asymptotic stability of the system (27) for the allocation vector $\vec{x}(t)$ fixed to an arbitrary feasible allocation vector $\vec{x}$ for every $t \geq 0$, which is of independent interest as it applies also for non-linear utility functions.

Proof. We first assume that $\vec{x}(t)$ is fixed to an arbitrary feasible allocation $\vec{x}$, for every $t \geq 0$, and then establish global asymptotic stability of the system (27), i.e. that every trajectory $\vec{w}(t)$ converges to a unique limit point. Let us use the notation $a_i = U_i'(x_i)x_i$, for every i. Let $L : \mathbb{R}_+^n \to \mathbb{R}$ be the function defined as follows

$$L(\vec{w}) = \sum_{j=1}^{n}\log(w_j) - \log(\sum_{j=1}^{n} w_j) - \sum_{j=1}^{n}\frac{w_j}{a_j}.$$

We show that for every trajectory $\vec{w}(t)$ of the system (27), we have $\frac{d}{dt}L(\vec{w}(t)) \geq 0$, for every $t \geq 0$. Indeed,

$$\frac{d}{dt}L(\vec{w}) = \sum_{i=1}^{n}\frac{\partial}{w_i}L(\vec{w})\frac{d}{dt}w_i = \sum_{i=1}^{n}\frac{1}{a_i}\left(\frac{1}{w_i} - \frac{1}{\sum_{j=1}^{n} w_j}\right)\left(a_i - \frac{w_i}{1-\frac{w_i}{\sum_{j=1}^{n} w_j}}\right)^2$$

which is non-negative for every $\vec{w} \in \mathbb{R}_+^n$. Now, for the system (27) there exists (a positive invariant set) S that is compact and such that if $\vec{w}(0) \in S$, then $\vec{w}(t) \in S$, for every $t \geq 0$. For example, it is not difficult to establish that one such set is $S = [0, \overline{w}]^n$, where $\overline{w} = \max_i\{\max\{v_i, w_i(0)\}\}$. By *LaSalle's theorem* (e.g. Theorem 4.4 [16]) we have that every solution $\vec{w}(t)$ started in S converges to the maximizer of the function $L(\vec{w})$, which is unique and given by $w_i = Ra_i/(a_i+R)$ where R is such that $\sum_i a_i/(a_i+R) = 1$.

Having established the above convergence, it is not difficult to observe that for every i,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T [u_i(t) - x_i p_i(\vec{x})]dt = x_i\frac{\partial}{\partial x_i}V(\vec{x})$$

and

$$\left|\left|\frac{1}{T}\int_0^T [u_i(t) - x_i p_i(\vec{x})]dt - x_i\frac{\partial}{\partial x_i}V(\vec{x})\right|\right| \leq c(T)$$

where $u_i(t) := v_i(w_i(t), R(\vec{w}(t)))$ and c is a strictly decreasing, continuous, and bounded function that converges to zero as T goes to infinity.

Finally, let $\vec{\overline{x}}(t)$, for $t \geq 0$, be the solution of the system $(d/dt)\overline{x}_i(t) = \overline{x}_i(t)\frac{\partial}{\partial x_i}V(\vec{\overline{x}}(t))$. Then, by the *averaging theorem* Theorem 10.5 [16], we have that if $\vec{\overline{x}}(\alpha t)$ is within the domain for every $0 \leq t \leq b/\alpha$ and $\vec{x}(0) - \vec{\overline{x}}(0) \in O(c(\alpha))$, then for every sufficiently small $\alpha > 0$, we have $\vec{x}(t) - \vec{\overline{x}}(\alpha t) = O(c(\alpha))$, for every $0 \leq t \leq b/\alpha$. This shows that the solution of the averaged dynamics of the system (26)-(27) approximates that of the system (23) and completes the proof. ∎

Same approach applies more generally for non-linear user utility functions but one would need to use an online estimator of the second derivative of the utility function in (25) from the observed bids submitted by users.[4] In

[4] Notice that the need to infer these second derivatives of the utility functions is not an artifact of the auction scheme that we use, but is intrinsic to any revenue maximization problem.

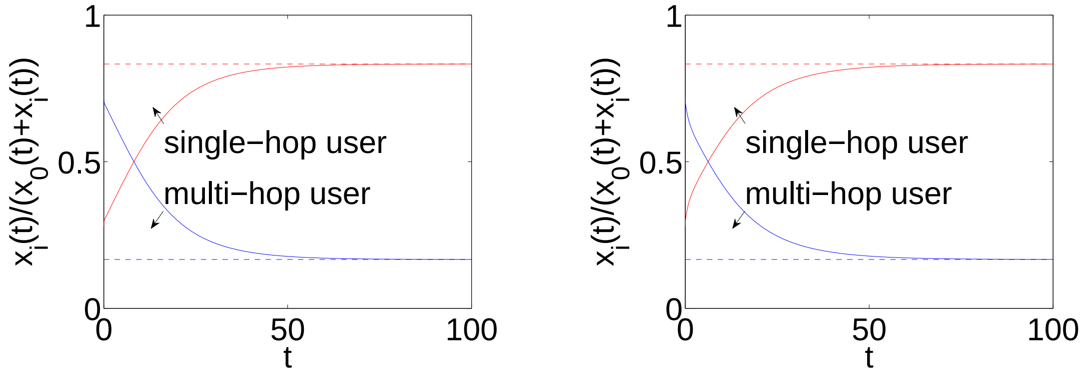

Figure 8: Convergence to equilibrium points for the parking-lot example: (Left) a priori known utility functions and (Right) a priori unknown utility functions.

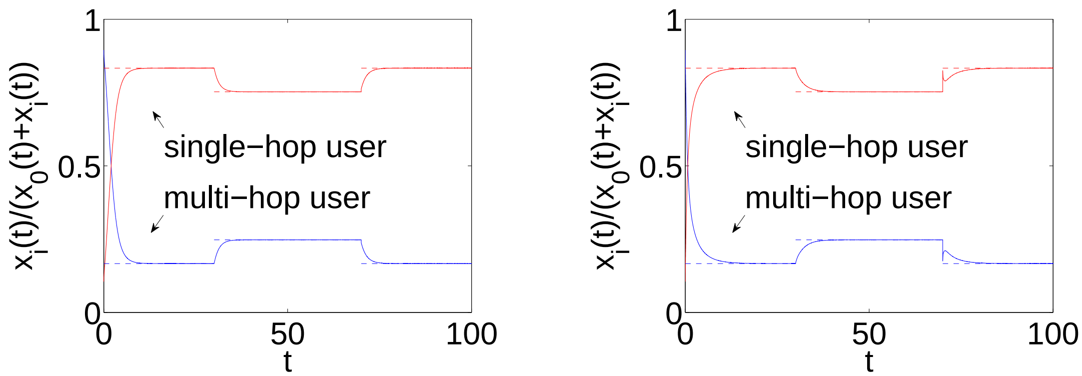

Figure 9: Another example for the convergence to equilibrium points for the parking-lot: (Left) a priori known utility functions and (Right) a priori unknown utility functions.

principle, this can be done by observing the effect of perturbing the allocation for a user on the bid submitted by this user, which we briefly discuss in the following. From (6) and the allocation rule $x_i = C_i w_i / \sum_j w_j$, we have $U_i'(x_i) = \frac{R}{x_i}\frac{w_i}{R}/(1-\frac{w_i}{R})$. Taking the derivative with respect to x_i, we obtain

$$U_i''(x_i)x_i = \frac{\frac{w_i}{R}}{1-\frac{w_i}{R}}\left(\frac{\partial}{\partial x_i}R - \frac{1}{x_i}R\right) + R\frac{1}{(1-\frac{w_i}{R})^2}\frac{\partial}{\partial x_i}\left(\frac{w_i}{R}\right).$$

Plugging this in (25), we have

$$\frac{\partial}{\partial x_i}R = \frac{R}{\sum_j \frac{w_j}{R}\left(1-\frac{w_j}{R}\right) - \frac{w_i}{R}\left(1-\frac{w_i}{R}\right)}\frac{\partial}{\partial x_i}\left(\frac{w_i}{R}\right).$$

Therefore, the gradient of the revenue can be fully expressed in terms of the bids $\vec{w}$ and the terms $(\partial/\partial x_i)(w_i/R)$, which can be estimated in an online fashion by perturbing the allocation of user i and observing the resulting change of w_i/R.

Parking-Lot Example. We demonstrate convergence of the iterative scheme (26)-(27) for the example of parking-lot network which we introduced in Section 3 (Figure 2). Recall that the resource consists of $n \geq 1$ links, each of capacity 1, with a multi-hop user 0 with allocation x_0 at each link and a single-hop user with allocation x_i at each link i. User utility functions are assumed to be linear $U_i(x) = v_i x$, for $x \geq 0$, where $v_i = v_1$, for $i = 1, 2, \ldots, n$, and $v_0, v_1 > 0$.

The Nash equilibrium allocation and the corresponding revenue are specified in the following lemma whose proof is simple and thus omitted.

Lemma 6 *Let* $\eta = \frac{n-1}{2\sqrt{n}}\sqrt{\frac{v_1}{v_0}}$. *For the parking-lot scenario, the Nash equilibrium allocation is* $1-x_1$ *for user* 0 *and* x_1 *for each user* $i = 1,2,\dots,n$, *where*

$$x_1 = \begin{cases} \frac{1}{2(1-\eta)}, & \textit{for } \eta < 1/2 \\ 1, & \textit{for } \eta \geq 1/2 \end{cases}. \tag{28}$$

The revenue at the Nash equilibrium is given by

$$R = \begin{cases} \frac{1}{4\eta(1-\eta)}(n-1)v_1, & \textit{for } \eta < 1/2 \\ (n-1)v_1, & \textit{for } \eta \geq 1/2. \end{cases}$$

We note that the Nash equilibrium allocation to a single-hop user is increasing with the ratio of the valuations v_1/v_0, from $1/2$ for $v_1/v_0 = 0$ to 1 for $v_1/v_0 = n/(n-1)^2$ and beyond this at each link the single-hop user is allocated the entire link capacity.

In the remainder of this section we illustrate convergence for two particular cases. In either case we consider the case with $n = 5$ links and linear utility functions specified by $v_0 = 5$ and $v_1 = v_2 = \cdots = v_n = 1$. The link cost functions are defined as

$$P_l'(x) = \begin{cases} 0, & 0 \leq x \leq \rho_0, \\ \left(\frac{1}{x} - \frac{1}{1-\rho_0}\right)^p, & \rho_0 < x \leq 1 \end{cases}$$

where $p > 0$ and in particular we use $p = 2$ and $\rho_0 = 0.8$. We show results for initial allocation $\vec{x}(0)$ such that $x_1(0) = x_2(0) = \cdots = x_n(0)$, so that due to symmetry $x_1(t) = x_2(t) = \cdots = x_n(t)$, for every $t \geq 0$. This simplifies the exposition. Indeed, we validated convergence for various other initial values other choices of other parameters but for space reasons confine to the above asserted setting.

We first demonstrate convergence for a closed system with a fixed set of users. In Figure 8 we show trajectories for the allocations $x_0(t)$ and $x_1(t)$ for both a priori known utility functions (left) and a priori unknown utility functions (right), with α set to 0.1. The results indeed validate convergence to the Nash equilibrium allocation, which are indicated with dashed lines. Our other example consists of an open system where at a specific time $t_1 \geq 0$ a single-hop user departs the system and then another such user arrives at time $t_2 > t_1$. In particular, we use the values $t_1 = 30$, $t_2 = 70$, and $\alpha = 0.8$. Figure 8 well validates convergence to the Nash equilibrium allocation in this case.

7 Conclusion

We considered a simple mechanism for allocation of a resource owned by a provider to users that allows for discrimination of users and general resource constraints specified by convex polyhedrons, thus allowing for provisioning of a variety of systems and services. We showed that in a competitive framework where everyone is selfish, in a wide set of cases, the mechanism guarantees nearly optimal revenue to the provider and competitive social efficiency (including a setting with multiple providers). Besides analysis of equilibrium points of the underlying competition game, we also showed how one would design an iterative algorithm that converges to the equilibrium points.

The work suggests several interesting directions for future research. First, it would be of interest to study revenue and efficiency properties for classes of user utility functions that are not accommodated by our framework, e.g. other than δ-utility functions. Second, it would also be of interest to purse a more in-depth analysis of convergence properties of the algorithms in this paper; in particular, it would be of interest to consider implementation details and address the stability properties of the underlying schemes.

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A Applications of General Polyhedral Environments

We explain some applications of the resource allocation problem with general polyhedral constraints.

Network bandwidth sharing

The most natural example is the bandwidth sharing game, where each provider owns a network of capacitated links, each user i is sending traffic along a path P_i and x_i is the data transfer rate for user i. In this case we have a resource constraint associated to each link e: $\sum_{i:e\in P_i} x_i \leq c_e$ where c_e is the capacity of link e.

Keyword auctions

The general convex constraints can also capture a general model of keyword auctions. The auction is for a single keyword, and there are n advertisers bidding to have their ad appear as a sponsored search result. There are finite set of outcomes, depending on which bidder gets displayed in which position. We describe each of these outcomes as a n dimensional vector whose coordinates are the expected number of click that the corresponding advertiser gets. More precisely, let $\vec{y}^1,\ldots,\vec{y}^N$ be all the possible outcome vectors, and $\vec{y}^k = (y^k_1,\ldots,y^k_n)$, where x^k_i is the expected number of clicks that advertiser i receives at outcome k. To think of keyword auctions as a convex resource allocation, we need to allow randomization in the allocation of bidders to positions. Choosing between the deterministic allocations by the probability distribution $\vec{p} = (p_1,\ldots,p_N)$, we have that $\sum_j p_j\vec{y}^j$ is the vector whose coordinates correspond to the expected number of clicks of an advertiser. Now the set of expected allocation vectors obtained this way is exactly the convex hull $conv(\vec{y}^1,\ldots,\vec{y}^N) = \{\vec{x} : \vec{x} = \sum_j p_j\vec{y}^j, p_j \in [0,1]$ for every j and $\sum_j p_j = 1\}$.

In this model, we will assume a natural condition on the externalities of the click-through rates: if we remove an ad from a position (by simply showing one fewer ad), the expected number of clicks received by the remaining ads does not decrease. Under this assumption, it is not hard to see that the set of all possible randomized allocation vectors, that is the convex hull $conv(\vec{y}^1,\ldots,\vec{y}^N)$, can be written as a polyhedron, which is exactly the constraint of the problem considered in this paper.

Traditional keyword auctions use an assignment of ads to positions (ordering) that depends on the bids times an individual weight, and do not use randomization [17, 3]. Furthermore, the most commonly used framework does not capture the externalities among ads. Our description of the outcomes associates an arbitrary nonnegative vector of click-through rates with each selection of bidders allocated to a positions. Therefore, this allows us to model the most general type of externalities between the ads shown. Externalities in keyword auctions are natural and important, for example, the valuation of a bidder for being, say, in position 2 depends on what ad is showed in position 1. For example, NIKE in position 1 makes position 2 less valuable for *sneakers* compared to having an unknown brand name in position 1.

To compete the description of the application in sponsored search auctions, we now give a formal definition and a proof about the externalities among advertisers in this applications.

We say that the a resource allocation problem with a feasible set of allocations $\mathcal{P}$ in $\mathbb{R}_+^n$ satisfies the *assumption of non-positive externalities* if for any allocation $\vec{x} \in \mathcal{P}$, and any coordinate k, there is an allocation $\vec{x}' \in \mathcal{P}$ such that (1) $x'_k = 0$, and (2) $x_i = 0$ implies $x'_i = 0$, and $x'_i \geq x_i$ for all $i \neq k$.

Claim 1 *The convex hull of some non-negative vectors $\vec{y}^1, \ldots, \vec{y}^N$ that satisfy the assumption of non-positive externalities can be written as $\{\vec{x} \in \mathbb{R}_+^n : Ax \leq \vec{b}\}$, for some matrix A and vector $\vec{b}$.*

Proof. Let C be the convex hull of $\vec{y}^1, \ldots, \vec{y}^N$. We will show that if these vectors satisfy the assumption of non-positive externalities, then for a vector $\vec{w} \in C$, and any vector $\vec{v}$ such that $0 \leq \vec{v} \leq \vec{w}$ is also in C. With this property, it is not difficult to see from basic convex geometry that the set C can be written as a polyhedron.

We prove this property by induction on the number of non-zero coordinates of $\vec{w}$. The claim is trivial when $w_i = 0$ for every i. Consider a vector $\vec{w} \in C$. The set C is the convex hull of $\vec{y}^1, \ldots, \vec{y}^N$, hence there exist non-negative real numbers $\alpha_1, \ldots, \alpha_N$ such that $\sum_i \alpha_i = 1$, and $w = \sum_i \alpha_i x^i$. Let k be the coordinate that minimizes the ratio v_i/w_i for $w_i \neq 0$, and let λ denote this ratio. By definition, $\lambda \leq 1$, and thus if $\lambda = 1$, there is nothing to prove. We use the definition of non-positive externalities for each $\vec{y}^i$ to obtain a vector $\vec{z}^i \in C$ with $z^i_k = 0$, and let $w' = \sum_i \alpha_i z^i$. Consider the vector $\lambda\vec{w} + (1-\lambda)\vec{w}' \in C$. By definition of λ, we have the following facts

- $\vec{w}' \in C$;
- $\vec{w}'$ has more zero coordinates than $\vec{w}$ (namely the kth coordinate);
- the kth coordinate of $\lambda\vec{w} + (1-\lambda)\vec{w}'$ is equal to v_k;
- $\lambda\vec{w} + (1-\lambda)\vec{w}' \geq \vec{v}$ (as $\vec{w}'_j \geq \vec{w}_j$ for all coordinates $j \neq k$);
- $\lambda\vec{w} \leq \vec{v}$ which follows as λ was the minimum ratio $\min_i v_i/w_i$.

The last two properties guarantee that there is a vector $0 \leq \vec{v}' \leq \vec{w}'$ such that $\lambda\vec{w} + (1-\lambda)\vec{v}' = \vec{v}$. Now, we use the induction hypothesis to $\vec{w}' \in C$ and $0 \leq \vec{v}' \leq \vec{w}'$ to show that $\vec{v}' \in C$, and hence, $\vec{v} = \lambda\vec{w} + (1-\lambda)\vec{w}' \in C$. ■

Scheduling jobs in data centers

This is a problem of allocating data center resources to users. In this application, typically each user needs to finish a job which requires reading many different blocks of data across machines in a data center. Let D^j_i be the amount of data of type j that job i needs to process. The utility function of each job i is related to its finishing time t_i, which is the maximum processing time of the job across all types of data that it requests. Equivalently, one can model that each job i tries to maximize the utility $U_i(x_i)$, where $x_i = 1/t_i$. In other words, x_i is the minimum among D^j_i/s^j_i, where s^j_i is the speed that job i can process data of type j. Typically, data centers are complex systems consisting of many clusters of machines and data has many copies across the clusters. The constraints on s^j_i are complex, but in many cases it can be captured by convex constraints. Therefore, the allocation vector $\vec{x}$ can also be captured by convex constraints. In this example, it is unrealistic to design a mechanism that requires every job to know exactly the complex constraints on $\vec{x}$. Simple mechanisms are crucial in these applications.

B Missing Proofs

B.1 Proof of Lemma 4

Items (i) and (ii) are straightforward to show. In the following, we show item (iii).

Let $h = f + g$. Given $a \geq 0$, let $b \geq 0$ be such that

$$[h'(a)a]' = h'(b). \tag{29}$$

We need to show that

$$h(b) - h'(b)b \leq \delta h(a)$$

which corresponds to

$$f(b) - f'(b)b + g(b) - g'(b)b \leq \delta(f(a) + g(a)). \tag{30}$$

Let b_1 and b_2 be such that

$$[f'(a)a]' = f'(b_1) \tag{31}$$
$$[g'(a)a]' = g'(b_2) \tag{32}$$

and, without loss of generality, assume $b_1 \leq b_2$.

Since f and g are δ-utilities, the following two relations hold

$$\begin{aligned} f(b_1) - f'(b_1)b_1 &\leq \delta f(a) \\ g(b_2) - g'(b_2)b_2 &\leq \delta g(a). \end{aligned}$$

Hence,

$$f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2 \leq \delta(f(a) + g(a)). \tag{33}$$

In view of (30) and (33), it suffices to show that

$$f(b) - f'(b)b + g(b) - g'(b)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2. \tag{34}$$

Note that $[h'(a)a]' = [f'(a)a]' + [g'(a)a]'$. Combining with (29), (31), and (32), we observe

$$f'(b_1) + g'(b_2) = f'(b) + g'(b).$$

Using this identity it is not difficult to conclude that $b_1 \leq b \leq b_2$ and that we can rewrite (34) as

$$f(b) - f'(b_1)b + g(b) - g'(b_2)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2.$$

The latter inequality indeed holds if the following two inequalities hold

$$\begin{aligned} f(b) - f(b_1) &\leq f'(b_1)(b - b_1) \\ g(b_2) - g(b) &\geq g'(b_2)(b_2 - b) \end{aligned}$$

but the latter two inequalities are indeed true as $b_1 \leq b \leq b_2$ and both f and g are concave functions.

B.2 Proof of the first part of Theorem 4

For both price taking users and price anticipating users, a Nash equilibrium is determined by a set of allocation vectors $(\vec{x}^1, \ldots, \vec{x}^m) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$. Consider the conventional best-response function

$$F : \mathcal{P}_1 \times \cdots \times \mathcal{P}_m \to \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$$

such that $(\vec{y}^1, \ldots, \vec{y}^m) = F(\vec{x}^1, \ldots, \vec{x}^m)$, where $\vec{y}^k$ is the allocation vector that maximizes the revenue for provider k, assuming other providers do not change their allocations. This mapping is continuous and thus by the fixed-point theorem, there exists an allocation vector where no provider k can increase his revenue by changing the allocation vector $\vec{x}^k$, which is a Nash equilibrium.

B.3 Proof of Lemma 5

By similar argument as in the proof of Theorem 2, we can assume that the convex set $\mathcal{P}_k$ is of the form

$$\sum_i \gamma_i x_i^k = 1, \text{ with } \gamma_i \geq 0 \text{ for each } i,$$

and we can derive the condition, for some $p > 0$,

$$\begin{aligned} \text{either } x_i &= 0 \\ \text{or } \frac{\frac{1}{\gamma_i}[U_i'(x_i)x_i^k]'}{(U_i'(x_i)x_i^k + R^k)^2} &= \frac{\frac{1}{\gamma_i}v_i^k}{(U_i'(x_i)x_i^k + R^k)^2} = p. \end{aligned} \tag{35}$$

Let us use the following notation, for each user i,

$$a_i = \frac{v_i^k}{\gamma_i} \text{ and } y_i = \frac{U_i'(x_i)x_i^k}{U_i'(x_i)x_i^k + R^k}. \tag{36}$$

Without loss of generality, we assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. From(36), we have

$$\begin{aligned} &\sum_i U_i'(x_i)x_i^k = R^k \sum_i \frac{y_i}{1-y_i} \geq R^k \left(\frac{y_1}{1-y_1} + \sum_{i \geq 2} y_i \right) = \\ &R^k \left(\frac{y_1}{1-y_1} + (1-y_1) \right) = R^k \frac{y_1^2 - y_1 + 1}{1-y_1} \end{aligned} \tag{37}$$

while

$$\max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i = \max_i a_i = a_1.$$

It is straightforward to observe that the following holds $U_i'(x_i)x_i^k = R^k \frac{y_i}{1-y_i}$. Therefore, for every i,

$$\gamma_i x_i^k = R^k \frac{\gamma_i}{U_i'(x_i)} \frac{y_i}{1-y_i}.$$

However,

$$a_i = \frac{[U_i'(x_i)x_i^k]'}{\gamma_i} \leq \frac{U_i'(x_i)}{\gamma_i}.$$

Therefore, we have

$$\gamma_i x_i^k = R^k \frac{\gamma_i}{U_i'(x_i)} \frac{y_i}{1-y_i} \leq R^k \frac{1}{a_i} \frac{y_i}{1-y_i}.$$

Thus,

$$a_1 = a_1 (\sum_i z_i) \gamma_i x_i^k \leq a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)}$$

We also have

$$\text{either } y_i = 0 \text{ or } \frac{a_i(1-y_i)^2}{(R^k)^2} = p > 0. \tag{38}$$

From the latter, the analysis follows the same steps as in the proof of Theorem 2, which yields the result

$$\sum_i U_i'(x_i)x_i^k \geq \frac{1}{1+2/\sqrt{3}} \max_{z \in \mathcal{P}_k} \sum_i v_i^k z_i.$$

C Efficiency in Competitive Environments

Lemma 7 *Admit same setting as in Theorem 2 and, in addition, assume that for at least k users* $v_i = \max_j v_j$. *Then, the efficiency is at least* $1 - \frac{1}{2k} + o(1/k)$.

Proof. Following the same steps as in the proof of Theorem 2, we have that the social welfare at the Nash equilibrium is at least

$$\frac{ky}{1-y} + (1-ky)$$

and the maximum social welfare is at most

$$\frac{1}{(1-y)^2}(ky(1-y)+1-ky)$$

for some $0 \leq y \leq 1/k$. It follows that the efficiency is at least

$$f_k(y) = (1-y)\left(\frac{2-y}{1-ky^2} - 1\right)$$

for some $0 \leq y \leq 1/k$. It remains only to establish that

$$\inf_{y \in [0,1/k]} f_k(y) = 1 - \frac{1}{2k} + o(1/k).$$

This follows by noting that for a minimizer y, $f_k'(y) = 0$, which is equivalent to

$$y^4 - \frac{5}{k}y^2 + \frac{2}{k}\left(2+\frac{1}{k}\right)y - \frac{2}{k^2} = 0.$$

Since $y \leq 1/k$, we neglect the term y^4 as it is of smaller order than other terms, which amounts to solving a quadratic equation whose solution in $[0,1/k]$ is given by

$$y = \frac{1}{5}\left(2+\frac{1}{k} - \sqrt{4-\frac{6}{k}+\frac{1}{k^2}}\right).$$

It readily follows that $y = \frac{1}{2k} + o(1/k)$ and plugging into $f_k(y)$ yields the asserted claim. ∎

D Relations with Other Concepts of Utilities

Relation to the efficiency of monopoly pricing. The dependency of the efficiency on the properties of the utility functions holds in general, not is not particular to our mechanism. Consider the classical case of monopoly pricing [25] – a single seller sells to buyers with utility function $U(x)$ and there is a constant marginal production cost $c > 0$. The seller optimizes the price p so as to maximize the profit. For given price p, buyers choose the quantity x that maximizes $U(x) - px$, so that $U'(x) = p$. The sellers finds the quantity x^m that maximizes $p(x)x - cx = U'(x)x - cx$, thus $[U'(x^m)x^m]' = c$. The social welfare is maximum at a quantity x^s that maximizes $U(x) - cx$, thus $U'(x^s) = c$. It follows that the efficiency is $(U(x^m) - cx^m)/(U(x^s) - cx^s)$, which in the geometric interpretation in Figure 10 corresponds to the ratio W'/L. Note that if the utility function is such that for some $\gamma \in [0,1]$, the efficiency of the monopoly pricing is greater or equal to γ for any marginal cost $c > 0$, then the utility function is a $\frac{1}{\gamma}$-utility.

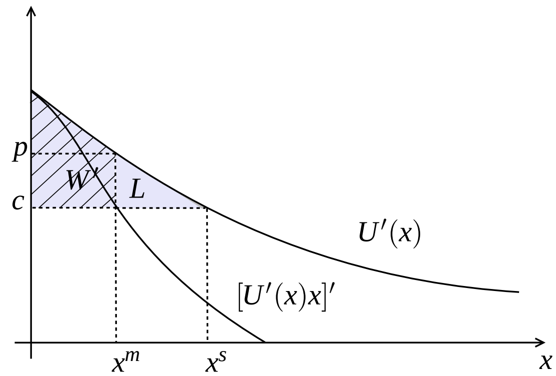

Figure 10: Efficiency for monopoly pricing: $\frac{W'}{L}$.

Relation to the elasticity of demand. It is natural to consider how the definition of δ-utility compares to existing characterizations of utility functions. A standard measure is the so-called *elasticity of demand* which is used to characterize the third-degree price discrimination [25, 24]. Let $p(x) = U'(x)$ where $p(x)$ is interpreted as the price at the output x, and the inverse function $x(p)$ is known as the demand function. The elasticity of demand at the output x is defined by $-(dx/x)/(dp/p)$. Note that this corresponds to $U'(x)/(-U''(x)x)$. Now, instead of considering the elasticity of demand at particular values of the output x, we consider the following uniform bound, let $\varepsilon > 0$ be such that

$$\frac{U'(x)}{-U''(x)x} \geq \varepsilon, \quad \text{for all } x \geq 0. \tag{39}$$

Note that in Figure 6, the left-hand side corresponds to the ratio of the length of the line segment $[(a,0),(a,U'(a))]$ and the length of the line segment $[(a,[U'(a)a]'),(a,U'(a))]$. Intuitively, we want this ratio to be large as we want the area L to be small relative to the area W. This indeed conforms to the fact that linear utility functions have infinite elasticity of demand, and by Lemma 9 we know that linear utility functions are 0-utilities.

The following lemma provides a relation between the elasticity of demand and δ-utility.

Lemma 8 *Suppose $U(x)$ is a non-negative utility function such that $U'(x)$ is non-increasing and concave and (39) holds. Then, $U(x)$ is a $\frac{2}{\varepsilon}$-utility function.*

Proof. For a concave function $U'(x)$, the area L in Figure 6 is less than or equal to the area of the triangle defined by the intersection of the lines $x = 0$, $y = [U'(a)a]'$, and the tangent to the function $U'(x)$ at $x = a$. The area of this triangle is $-2U''(a)a^2$. Hence,

$$\frac{L}{W} \leq \frac{-2U''(a)a^2}{U(a)} = \frac{-2U''(a)a}{U'(a)}\frac{U'(a)a}{U(a)} \leq \frac{-2U''(a)a}{U'(a)} \leq \frac{2}{\varepsilon}$$

where the first inequality follows as $U'(a)a \leq U(a)$ holds for any non-negative concave utility function $U(a)$ (which holds as $U'(a)$ is assumed to be non-increasing) and the last inequality is by (39). ■

E A Catalogue of Utility Functions

The following lemma shows that many utility functions found in literature are δ-utilities.

Lemma 9 *We have the following properties:*

(i) $U(x) = \alpha x$, *for* $\alpha > 0$, *or a truncated linear function*[5], *is a* 0*-utility;*

(ii) $U(x)$ *such that* $U'(x)$ *is a concave function is a* 2*-utility;*

[5]That is, $U(x) = \min\{\alpha x, y\}$, for every $x \geq 0$, for some $y > 0$.

(iii) $U(x) = \log\left(\frac{c+x}{c}\right)$, *for* $c > 0$, *is a* 2*-utility;*

(iv) $U(x) = (c+x)^\alpha$ *for* $c \geq 0$ *and* $0 \leq \alpha \leq 1$ *is a* $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$*-utility for* $0 \leq \alpha \leq \frac{1}{2}$, *and a* 1*-utility for* $\frac{1}{2} < \alpha \leq 1$*; simpler but weaker, it is a* $\frac{e}{2}$*-utility for* $0 < \alpha \leq 1$.

(v) $U(x) = \frac{w^\alpha}{1-\alpha}(c+x)^{1-\alpha}$, *for* $\alpha \in [0,1) \cup (1,\infty)$, *and* $U(x) = w\log(c+x)$, *for* $\alpha = 1$, *with* $w > 0$ *and any* $c \geq 0$, *is a* 1*-utility for* $0 \leq \alpha \leq \frac{1}{2}$ *and a* $\alpha(1-\alpha)^{-\frac{1-\alpha}{\alpha}}$*-utility for* $\frac{1}{2} < \alpha < 1$*; simpler but weaker, it is a* $\frac{e}{2}$*-utility for* $0 \leq \alpha < 1$.

(vi) $U(x) = \alpha \cdot \arctan\left(\frac{x}{\alpha}\right)$, *for* $\alpha > 0$, *is a* 2*-utility.*

Proof. We show proofs for each item in the following.

Item (i)

It suffices to consider truncated linear functions, i.e. for $\alpha > 0$ and $y > 0$, $U(x) = \min\{\alpha x, y\}$, $x \geq 0$, as linear functions are a special case with $y = \infty$. Clearly, we have $U(b) - U'(b)b = 0$, for any $b \geq 0$, hence $\delta = 0$.

Item (ii)

Consider the tangent to $U'(x)$ at the point $x = a$; see Figure 11. This tangent forms the triangle BDF. Note that the area L is less or equal to the area of the triangle BDF. The side DF of the triangle is of length $-2U''(a)a$. The side FB of the triangle is of length $2a$. Hence, the area of the triangle is equal to $-2U''(a)a^2$. Now, note that the area W is greater or equal to the area of the rectangle $ACEF$. The sides of this rectangle are of length $-U''(a)a$ and a. Hence, the area of the rectangle is $-U''(a)a^2$. It follows that $L/W \leq 2$.

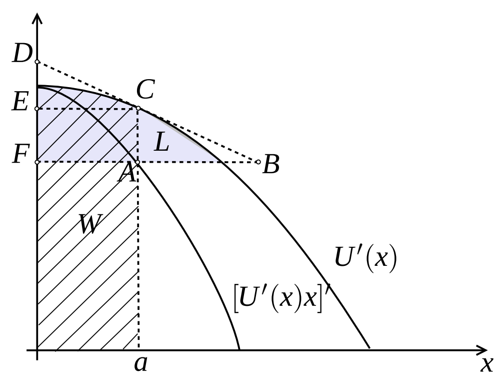

Figure 11: $U'(x)$ concave.

Item (iii)

We have

$$U'(x) = \frac{1}{c+x} \text{ and } [U'(x)x]' = \frac{c}{(c+x)^2}.$$

From $U'(b) = [U'(a)a]'$ we have

$$U'(b) = \frac{1}{c+b} = \frac{c}{(c+a)^2}$$

and

$$b = \frac{(c+x)^2}{c} - c.$$

It follows

$$\begin{aligned}\frac{U(b)-U'(b)b}{U(a)} &= \frac{2\log\left(\frac{c+a}{c}\right)+\left(\frac{c}{c+a}\right)^2-1}{\log\left(\frac{c+a}{c}\right)} \\ &= \frac{2\log(u)-u^2+1}{\log(u)} = 2-\frac{u^2-1}{\log(u)} := \varphi(u)\end{aligned}$$

where $u = c/(c+a)$. Since $(u^2-1)/\log(u) \geq 0$, we have $\varphi(u) \leq 2$, for all $u \in [0,1]$. This bound is tight; achieved at $u = 0$.

Item (iv)

We have

$$U(x) = (c+x)^{\alpha} \tag{40}$$

$$U'(x) = \alpha(c+x)^{\alpha-1} \tag{41}$$

$$[U'(x)x]' = \alpha(c+x)^{\alpha-1}\left[1-(1-\alpha)\frac{x}{c+x}\right]. \tag{42}$$

It follows

$$\begin{aligned}\frac{U(b)-U'(b)b}{U(a)} &= (1-\alpha)\left[1-(1-\alpha)\frac{a}{c+a}\right]^{-\frac{\alpha}{1-\alpha}}+ \\ &+\alpha\frac{c}{c+a}\left[1-(1-\alpha)\frac{a}{c+a}\right].\end{aligned} \tag{43}$$

Remark Note that for $c = 0$, we have

$$\frac{U(b)-U'(b)b}{U(a)} = (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$$

which is independent of $a \geq 0$.

Let us consider the right-hand side with the following change of variables $u = a/(c+a)$,

$$f_{\alpha}(u) := (1-\alpha)\left[1-(1-\alpha)u\right]^{-\frac{\alpha}{1-\alpha}}+\alpha(1-u)\left[1-(1-\alpha)u\right].$$

It is not difficult to note that $f'_{\alpha}(u)$ is non-decreasing on $[0,1]$, hence $f_{\alpha}(u)$ is a convex function on $[0,1]$. It follows that the function $f_{\alpha}(u)$ over $u \in [0,1]$ achieves maximum at either $u = 0$ or $u = 1$, with values $f_{\alpha}(0) = 1$ and $f_{\alpha}(1) = (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$. We claim

$$\max_{u\in[0,1]} f_{\alpha}(u) = \begin{cases} (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}} & 0 \leq \alpha \leq \frac{1}{2} \\ 1 & \frac{1}{2} < \alpha < 1. \end{cases}$$

Indeed, $f_{\alpha}(0) \leq f_{\alpha}(1)$ if and only if $\alpha^{\alpha} \leq (1-\alpha)^{1-\alpha}$. The function x^x is non-decreasing, thus the last inequality holds if and only if $\alpha \leq 1-\alpha$, i.e. $\alpha \leq 1/2$. The claim follows.

We now show that $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}} \leq \frac{e}{2}$, for all $\alpha \in [0,1]$. Indeed, the function $f(\alpha) := (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ achieves the maximum value at the same points as the function $g(u) = \log f(\alpha)$. We have

$$g(\alpha) = \log(1-\alpha) - \frac{\alpha}{1-\alpha}\log(\alpha).$$

It is straightforward to obtain

$$g'(\alpha) = \frac{1}{1-\alpha}\left[2+\frac{1}{1-\alpha}\log(\alpha)\right].$$

At a point α^* at which $g(\alpha)$ is maximum, we have $g'(\alpha^*) = 0$, which is equivalent to

$$\alpha^* = e^{-2(1-\alpha^*)}.$$

It follows

$$f(\alpha^*) = (1-\alpha^*)e^{2\alpha^*} = (1-\alpha^*)e^{-2(1-\alpha^*)}e^2 \leq$$
$$\leq e^2 \max_{x\in[0,1]} xe^{-2x} = \frac{e}{2}.$$

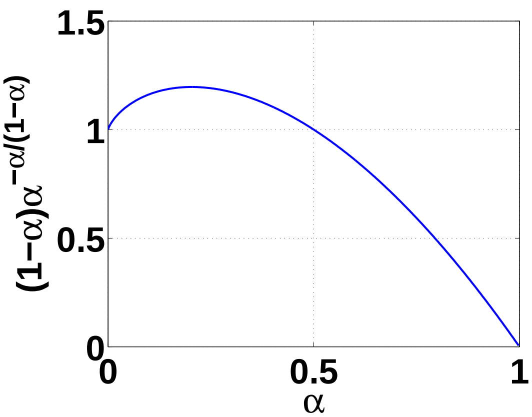

Figure 12: The function $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ versus α.

Remark. The result establishes a uniform bound that holds for any $0 \leq \alpha < 1$, equal to $e/2 \approx 1.359$. This bound is not tight and can be improved, albeit slightly. Figure 12 shows the function $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$. Finding the maximum value numerically, we obtain the value 1.196.

Item (v)

We have, for $\alpha \geq 0$,

$$U'(x) = \left(\frac{w}{c+x}\right)^{\alpha}.$$

It is not difficult to check that the function $U'(x)x$ is a concave function only if $0 \leq \alpha \leq 1$. Therefore, in the following we consider $\alpha \in [0,1]$.

Case 1: $\alpha \in [0,1)$. By straightforward calculus, we have

$$\frac{U(b)-U'(b)b}{U(a)} = \alpha\left[1-\alpha\frac{a}{c+a}\right]^{-\frac{1-\alpha}{\alpha}} + (1-\alpha)\frac{c}{c+a}\left[1-\alpha\frac{a}{c+a}\right].$$

Note that this is the same as (43) in Section E, but α replaced with $1-\alpha$, hence the results in Section E apply by replacing α with $1-\alpha$. We obtain that $\delta = 1$ for $0 \leq \alpha \leq \frac{1}{2}$, and $\delta = \alpha(1-\alpha)^{-\frac{1-\alpha}{\alpha}}$, for $\frac{1}{2} < \alpha < 1$. From the analysis above, we can take $\delta = e/2$ for $\frac{1}{2} < \alpha < 1$.

Case 2: $\alpha = 1$. Consider the case $c = 0$. Note that $U'(b) = [U'(a)a]'$ is equivalent to

$$\frac{w}{b} = 0.$$

Thus, $b = \infty$, from which it follows

$$\frac{U(b)-U'(b)b}{U(a)} = \frac{\log(b)-1}{\log(a)} = \infty, \text{ for any finite } a > 0.$$

This shows that $\delta = \infty$, and hence for $\alpha = 1$, $U(x)$ is not a δ-utility.

Item (vi)

We have

$$U'(x) = \frac{\alpha^2}{\alpha^2 + x^2} \text{ and } [U'(x)x]' = \alpha^2 \frac{\alpha^2 - x^2}{(\alpha^2 + x^2)^2}.$$

Note that $[U'(x)x]' \geq 0$ if and only if $x \leq \alpha$, and $[U'(x)x]'$ is non-increasing in $[0,\alpha]$, hence, condition (i) of Definition 2 is verified with $x_0 = \alpha$.

From $U'(b) = [U'(a)a]'$ it follows

$$U'(b) = \alpha^2 \frac{\alpha^2 - a^2}{(\alpha^2 + a^2)^2} \text{ and } b = a\sqrt{\frac{3\alpha^2 + a^2}{\alpha^2 - a^2}}.$$

We need to show that

$$\varphi_\alpha(a) := \frac{U(b(a)) - U'(b(a))b(a)}{U(a)} \leq 2, \text{ for all } a \in [0,\alpha]$$

where

$$\varphi_\alpha(a) = \frac{\alpha \cdot \arctan\left(\frac{a}{\alpha}\sqrt{\frac{3\alpha^2+a^2}{\alpha^2-a^2}}\right) - a\alpha^2 \frac{\sqrt{(\alpha^2-a^2)(3\alpha^2+a^2)}}{(\alpha^2+a^2)^2}}{\alpha \cdot \arctan\left(\frac{a}{\alpha}\right)}.$$

Note that $\varphi_\alpha(a) = \varphi_1(a/\alpha)$, hence it suffices to consider $\varphi_1(a)$ over $[0,1]$.

Condition $\varphi_1(a) \leq 2$, for $a \in [0,1]$, can be rewritten as

$$\arctan\left(a\sqrt{\frac{3+a^2}{1-a^2}}\right) - 2\arctan(a) \leq$$

$$\leq a\frac{\sqrt{(1-a^2)(3+a^2)}}{(1+a^2)^2}, \; a \in [0,1].$$

Clearly, the right-hand side is greater or equal to zero for all $a \in [0,1]$. The claim follows by noting that the left-hand side is less than equal to zero for all $a \in [0,1]$. To see this, note

$$\begin{aligned} g(a) &:= \arctan\left(a\sqrt{\frac{3+a^2}{1-a^2}}\right) - 2\arctan(a) \\ &= \arctan\left(a\sqrt{\frac{3+a^2}{1-a^2}}\right) - \arctan\left(\frac{2a}{1-a^2}\right) \end{aligned}$$

where equality follows from the elementary identity $\arctan(x) = 2\arctan\left(\frac{x}{1+\sqrt{1+x^2}}\right)$. Hence, $g(a) \leq 0$ if and only if

$$a\sqrt{\frac{3+a^2}{1-a^2}} \geq \frac{2a}{1-a^2}$$

but this can be rewritten as $(1-a^2)^2 \geq 0$, hence the proof. It can be easily checked that the equality in $\varphi_1(a) \leq 2$ is achieved for $a = 1$, hence $\delta = 2$ is tight.