Quantum MDS Codes over Small Fields

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Abstract—We consider quantum MDS (QMDS) codes for quantum systems of dimension q with lengths up to $q^2 + 2$ and minimum distances up to $q + 1$. We show how starting from QMDS codes of length $q^2 + 1$ based on cyclic and constacyclic codes, new QMDS codes can be obtained by shortening. We provide numerical evidence for our conjecture that almost all admissible lengths, from a lower bound $n_0(q, d)$ on, are achievable by shortening. Some additional codes that fill gaps in the list of achievable lengths are presented as well along with a construction of a family of QMDS codes of length $q^2 + 2$, where $q = 2^m$, that appears to be new.

Keywords—quantum error correction, quantum MDS codes

I. INTRODUCTION

QUANTUM error-correcting codes (QECC) are a key ingredient to implement information processing based on quantum mechanics. For quantum systems composed of n subsystems of dimension $q \geq 2$, so-called *qudits*, a quantum code $\mathcal{C} = ((N, K))_q$ is a k-dimensional subspace of the Hilbert space $(\mathbb{C}^q)^{\otimes n}$. If the dimension of the code $\mathcal{C}$ is q^k, it will be denoted by $\mathcal{C} = [\![n, k, d]\!]_q$, where d is the minimum distance. A code with minimum distance d is able to correct errors that affect no more than $(d-1)/2$ of the subsystems. The quantum Singleton bound [13], [17] relates the parameters n, k, and d as follows:

$$n + 2 \geq k + 2d \tag{1}$$

A quantum code for which equality holds in (1) is called a quantum MDS (QMDS) code. For classical codes, the existence of an MDS code $C = [n, k, n + 1 - k]_q$ implies the existence of MDS codes $C' = [n', k', n' + 1 - k']_q$ for all $k' \leq k$, $n' \leq n$, $k \leq n'$. For quantum codes, this is not true in general, i.e., a QMDS code $\mathcal{C} = [\![n, n + 2 - 2d, d]\!]_q$ does not necessarily imply the existence of QMDS codes of smaller length or smaller dimension.

For any number of qudits, the full space is a trivial QMDS code $\mathcal{C} = [\![n, n, 1]\!]_q$, where $q > 1$ can be any integer, not necessarily a prime power. QMDS codes with distance $d = 2$ exist for even length n when $q = 2$ (see [18]), and for all lengths $n \geq 2$ when $q > 2$ is a prime power (see below). This implies the existence of QMDS codes with $d = 2$ for all lengths $n \geq 2$ when q is odd or divisible by 4.

When the length of the code is bound by $n \leq q$, QMDS codes can be obtained from extended Reed-Solomon codes (see, e.g., [7]). Single-error-correcting QMDS codes for length $4 \leq n \leq q^2 + 1$ have been discussed in [14] for odd prime powers q, and more generally in [6], [10].

Quantum MDS codes of length $n \in \{q^2 - 1, q^2, q^2 + 1\}$ have been discussed in [10], [19]. In [10] there are also QMDS

codes of length n in the range $q+1 < n < q^2-1$, with the minimum distance d bounded by $d \leq (q+5)/4$. In [20], QMDS codes for certain lengths in the range $q+1 < n < q^2-1$ were constructed based on generalized Reed-Muller codes.

More recently, QMDS codes with a larger range for the minimum distance based on cyclic and constacyclic codes have been derived (see, e.g., [4], [11], [21], [22]). Those constructions put some constraints on the length n of the code, e.g., n has to be a divisor of $q^2 \pm 1$, and in most cases, the minimum distance is bounded by some fraction of q.

Here, extending our results from [19], we show that QMDS codes exists for essentially all lengths n in the range $n_0(q, d) \leq q^2 + 1$, where the lower bound n_0 grows with the minimum distance d. For most of these QMDS codes, the minimum distance is bounded by $d \leq q+1$, but we present also some examples of qutrit and ququad QMDS codes exceeding this bound.

After recalling basic results about stabilizer codes and construction of classical MDS codes in Section III, Section IV presents the main technique how shorter QMDS codes with the same minimum distance can be obtained. Theoretical results are summarized in Section V, supplemented by computational results in Section VI. All computations have been performed using the computer algebra system Magma [2].

II. STABILIZER CODES

Most quantum error-correcting codes are so-called stabilizer codes. Here we briefly summarize the basic results which are relevant in our context (for more details, see e.g. [1], [3], [12]).

The construction of stabilizer codes is based on classical codes which are self-orthogonal with respect to a symplectic inner product. The most general construction of a stabilizer code for qudits starts with an additive code $C = (n, p^\ell)_{q^2}$ of length n over the a quadratic extension field $\mathbb{F}_{q^2}$. Note that the code does not need to be $\mathbb{F}_q$- or $\mathbb{F}_{q^2}$-linear, but just $\mathbb{F}_p$-linear, where $q = p^m$ and p is prime.

Here we consider only the special case that the code is $\mathbb{F}_{q^2}$-linear. In this case, the symplectic inner product is equivalent to the so-called Hermitian inner product. For vectors $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{F}_{q^2}^n$, it is defined as

$$\boldsymbol{c} * \boldsymbol{c}' = \sum_{i=1}^{n} c_i^q c_i'. \tag{2}$$

We consider the dual code with respect to this Hermitian inner product.

Definition 1 (Hermitian dual code): Given a linear code $C = [n, k]_{q^2}$ over $\mathbb{F}_{q^2}$, the Hermitian dual C^* is given by

$$C^* = \{\boldsymbol{v} \colon \boldsymbol{v} \in \mathbb{F}_{q^2}^n \mid \forall \boldsymbol{c} \in C \colon \boldsymbol{c} * \boldsymbol{v} = 0\}. \qquad (3)$$

The Hermitian dual code $C^* = [n, n - k]_{q^2}$ is an $\mathbb{F}_{q^2}$-linear code of dimension $n - k$.

The following proposition is a central result relating classical codes and quantum stabilizer codes (see, e.g., [12, Corollary 19]).

Proposition 2: Let $C = [n, n - k]_{q^2}$ be an $\mathbb{F}_{q^2}$-linear code that is contained in its Hermitian dual $C^* = [n, k, d^*]_{q^2}$. Then there exists a quantum stabilizer code $\mathcal{C} = [\![n, 2k - n, d]\!]_q$. The minimum distance d is given by

$$d = \min\{\mathrm{wgt}\ \boldsymbol{c} \colon \boldsymbol{c} \in C^* \setminus C\} \geq d^*. \qquad (4)$$

If equality holds in (4), the code is said to be *pure*.

In [17] it has been shown that a QMDS code is always pure.

Shortening of the self-orthogonal code C yields the following derivation rule (see also [12, Lemma 70]).

Proposition 3: Assume that there is a pure stabilizer code $\mathcal{C} = [\![n, k, d]\!]_q$ with $d > 1$. Then there exists a QECC $\mathcal{C}' = [\![n - 1, k + 1, d - 1]\!]_q$.

Proof: (sketch) When we puncture the code C^* corresponding to $\mathcal{C}$ at say the first position, we obtain a code C'^* of length $n - 1$ which has the same number of codewords as C^* and minimum distance $d' \geq d - 1$. The code $(C'^*)^* = C'$ contains all vectors $\boldsymbol{c}'$ for which $0\boldsymbol{c}' \in C$. Hence $C' \subset C'^*$. The dimension of C' is one less than the dimension of C, resulting in an increase of the dimension of the quantum code by one. ∎

Note that shortening of the code C^* corresponds to puncturing the code C. However, after puncturing, the code C' need no longer be self-orthogonal with respect to the symplectic inner product.

Repeated application of Proposition 3 yields the following.

Corollary 4: Assume that a QMDS code $\mathcal{C} = [\![n, n + 2 - 2d, d]\!]_q$ exists. Then for all $0 \leq s < d$, there exist also QMDS codes $\mathcal{C}' = [\![n - s, n + s + 2 - 2d, d - s]\!]_q$.

III. CLASSICAL MDS CODES

In order to construct quantum MDS codes of length $q^2 + 1$, we start with cyclic or constacyclic MDS codes (see also [8], [19]). In order to simplify the notation, without loss of generality, we consider codes over the field $\mathbb{F}_q$ instead of the field $\mathbb{F}_{q^2}$.

Theorem 5: For any k, $1 \leq k \leq q + 1$, there exists a $[q + 1, k, q - k + 2]_q$ MDS code over $\mathbb{F}_q$ that is either cyclic or constacyclic.

Proof: Let ω denote a primitive element of $\mathbb{F}_{q^2}$. Hence $\alpha := \omega^{q-1}$ is a primitive $(q + 1)$-th root of unity.

First we consider the case when $q + 1 - k$ is odd. We define the following polynomial of degree $2\mu + 1$:

$$g_1(z) := \prod_{i=-\mu}^{\mu} (z - \alpha^i). \qquad (5)$$

Its zeros α^i and α^{-i} are conjugates of each other since $\alpha^q = \alpha^{-1}$. Hence $g_1(z)$ a polynomial over $\mathbb{F}_q$. The resulting cyclic code C over $\mathbb{F}_q$ has length $q + 1$ and dimension $q - 2\mu$. The generator polynomial $g_1(z)$ has $2\mu + 1$ consecutive zeros, so the BCH bound yields $d \geq 2\mu + 2$. Therefore C is an MDS code $[q + 1, q - 2\mu, 2\mu + 2]_q$.

If $q + 1 - k$ is even and q is even too, the polynomial

$$g_2(z) := \prod_{i=q/2-\mu}^{q/2+1+\mu} (z - \alpha^i)$$
$$= \prod_{i=q/2-\mu}^{q/2} (z - \alpha^i)(z - \alpha^{-i}) \qquad (6)$$

has degree $2\mu + 2$. It is a polynomial over $\mathbb{F}_q$ with $2\mu + 2$ consecutive zeros, so the resulting code is an MDS code with parameters $[q + 1, q - 1 - 2\mu, 2\mu + 3]_q$.

Finally, if $q + 1 - k$ is even and q is odd, consider the polynomial

$$g_3(z) := \prod_{i=1}^{\mu} (z - \omega\alpha^i)(z - \omega\alpha^{1-i}) \qquad (7)$$

of degree 2μ. The roots $\omega\alpha^i$ and $\omega\alpha^{1-i}$ are conjugates of each other as $(\omega\alpha^i)^q = \omega^{(1+(q-1)i)q} = \omega^{q+(1-q)i} = \omega^{1+(q-1)(1-i)} = \omega\alpha^{1-i}$, so $g_3(z)$ is a polynomial over $\mathbb{F}_q$. Furthermore, $g_3(z)$ divides $z^{q+1} - \omega^{q+1} \in \mathbb{F}_{q^2}[z]$ as $(\omega\alpha^i)^{q+1} = \omega^{q+1}$. Therefore $g_3(z)$ defines a constacyclic code C of length $q + 1$ and dimension $q + 1 - 2\mu$ over $\mathbb{F}_q$. From the analogue of the BCH bound for constacyclic codes (see, e.g., [16]), we have $d \geq 2\mu + 1$. Hence C is an MDS code with parameters $[q + 1, q + 1 - 2\mu, 2\mu + 1]_q$. ∎

Remark 6: Theorem 5 is a slightly modified version of Theorem 9 in [15, Ch. 11, §5]. There only cyclic codes are considered; the construction fails when both q and k are odd (see also the preface to the third printing of [15]).

IV. SHORTENING QUANTUM CODES

While classical linear codes can be shortened to any length, i.e., from a code $[n, k, d]$ one obtains a code $[n - r, k' \geq k - r, d' \geq d]$ for any r, $0 \leq r \leq k$, this is in general not true for quantum codes. However, in [17] it is shown how quantum codes can be shortened using the so-called puncture code. Here we recall the main results for $\mathbb{F}_{q^2}$-linear codes.

Definition 7 (puncture code): Let $C = [n, k]_{q^2}$ be an $\mathbb{F}_{q^2}$-linear code. The puncture code of C is defined as

$$P(C) := \left\langle \{(c_i^q c_i')_{i=1}^n \colon \boldsymbol{c}, \boldsymbol{c}' \in C\right\rangle^{\perp} \cap \mathbb{F}_q^n, \qquad (8)$$

where the angle brackets denote the $\mathbb{F}_{q^2}$-linear span.

From [17, Theorem 3] we get:

Theorem 8: Let $C = [n, k]_{q^2}$ be an $\mathbb{F}_{q^2}$-linear code, not necessarily self-orthogonal, of length n and dimension k such that the Hermitian dual code $C^* = [n, n - k]_{q^2}$ has minimum distance d. If there exists a codeword in $P(C)$ of weight r, then there exists a pure QECC $[\![r, k', d']\!]_q$ for some $k' \geq r - 2k$ and $d' \geq d$.

Proof: Let $\boldsymbol{x} \in P(C)$ be a codeword of weight r and let $S = \{i\colon i \in \{1, \ldots, n\} \mid x_i \neq 0\}$ denote its support. Note that the norm $\mathcal{N}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x) = x^{q+1}$ is surjective. Hence there exists a vector $\boldsymbol{y} \in \mathbb{F}_{q^2}^n$ such that $y_i^{q+1} = x_i$ for $1 \leq i \leq n$. We define the code $\widetilde{C}$ to be

$$\widetilde{C} := \left\{ (y_i c_i)_{i=1}^n \colon \boldsymbol{c} \in C \right\}, \tag{9}$$

i, e., we pointwise multiply the codewords by the corresponding elements of $\boldsymbol{y}$. For arbitrary $\widetilde{\boldsymbol{c}}, \widetilde{\boldsymbol{c}}' \in \widetilde{C}$, we get

$$\widetilde{\boldsymbol{c}} * \widetilde{\boldsymbol{c}}' = \sum_{i=1}^n \widetilde{c}_i^q \widetilde{c}_i' = \sum_{i=1}^n (y_i c_i)^q y_i c_i' = \sum_{i=1}^n x_i c_i^q c_i'. \tag{10}$$

From (8) it follows that (10) vanishes, i.e., $\widetilde{C}$ is self-orthogonal. As (10) depends only on the coordinates of $\boldsymbol{x}$ that are non-zero, we can delete the other positions in $\widetilde{C}$ and obtain an $\mathbb{F}_{q^2}$-linear self-orthogonal code $D \subseteq \mathbb{F}_{q^2}^r$ given by

$$D := \left\{ (y_i c_i)_{i \in S} \colon \boldsymbol{c} \in C \right\}.$$

Puncturing the code $\widetilde{C}$ may reduce its dimension. Hence D has parameters $D = [n, \widetilde{k}]_{q^2}$ for some $\widetilde{k} \leq k$. The dual code D^* is obtained by shortening the code C^*, and multiplying the resulting codewords by the corresponding non-zero entries of $\boldsymbol{y}$. Hence the minimum distance d' of D^* is not smaller than the minimum distance of C^*. This shows $d' \geq d$. Overall, we get a quantum code with parameters $\mathcal{C}' = [\![r, k', d']\!]_q$, where $k' = r - 2\widetilde{k} \geq r - 2k$. ∎

It should be stressed that the puncture code $P(C)$ can be computed for any code C, not only for self-orthogonal ones. In particular, using a codeword of maximal weight in $P(C)$, an arbitrary linear code can be converted into a self-orthogonal code.

The following obvious lemma will prove useful:

Lemma 9: If $C_1 \subseteq C_2$, then $P(C_2) \subseteq P(C_1)$.

For cyclic or constacyclic linear codes over $\mathbb{F}_{q^2}$, the puncture code will be again cyclic or constacyclic, respectively. We have the following characterization:

Theorem 10: Let $C^* = [n, k]_{q^2}$ be an $\mathbb{F}_{q^2}$-linear cyclic code with defining set $\mathcal{Z}$, i.e., the generator polynomial $g(x)$ of C^* has roots $\{\alpha^i\colon i \in \mathcal{Z}\}$ where α is a primitive n-th root of unity. For a constacyclic code $C^* = [n, k]_{q^2}$ with shift constant β^n, the generator polynomial $g(x)$ is a divisor of $x^n - \beta^n$, and its roots can be expressed as $\{\beta\alpha^i\colon i \in \mathcal{Z}\}$.

Then the puncture code $P(C)$ is a (consta)cyclic code over $\mathbb{F}_q$ with defining set

$$\mathcal{Z}' = \{iq + jq^2\colon i, j \in \mathcal{Z}\}. \tag{11}$$

Proof: First note that a cyclic code is a constacyclic code with shift constant $\beta^n = 1$. A parity check matrix for a (consta)cyclic code is given by

$$H = \left((\beta\alpha^i)^0, (\beta\alpha^i)^1, \ldots, (\beta\alpha^i)^{n-1} \right)_{i \in \mathcal{Z}}. \tag{12}$$

For an $\mathbb{F}_{q^2}$-linear code, the symplectic dual code equals the Hermitian dual code, which is the code obtained by Galois

conjugation of the usual dual code. Therefore, a generator matrix of C is given by

$$\begin{aligned} G &= \left((\beta\alpha^i)^0, (\beta\alpha^i)^q, \ldots, (\beta\alpha^i)^{(n-1)q} \right)_{i \in \mathcal{Z}} \\ &= \left((\beta^q\alpha^i)^0, (\beta^q\alpha^i), \ldots, (\beta^q\alpha^i)^{n-1} \right)_{i \in \mathcal{Z}^q}, \end{aligned} \tag{13}$$

where $\mathcal{Z}^q = \{i^q\colon i \in \mathcal{Z}\}$. From (8) it follows that a parity check matrix of $P(C)$ is given by the component-wise product of the rows of G and their Galois conjugates:

$$\begin{aligned} &H_{P(C)} \\ &= \left((\beta^q\alpha^i)^0 (\beta^{q^2}\alpha^{qj})^0, \ldots (\beta^q\alpha^i)^{n-1}(\beta^{q^2}\alpha^{qj})^{n-1} \right)_{i,j \in \mathcal{Z}^q} \\ &= \left((\tilde{\beta}\alpha^{i+qj})^0, (\tilde{\beta}\alpha^{i+qj}), \ldots (\tilde{\beta}\alpha^{i+qj})^{n-1} \right)_{i,j \in \mathcal{Z}^q}, \end{aligned} \tag{14}$$

where $\tilde{\beta} = \beta^{q(q+1)}$. Note that $\beta^n \in \mathbb{F}_{q^2}$, and hence $\tilde{\beta}^n = (\beta^n)^{q(q+1)} = (\beta^n)^{q+1} \in \mathbb{F}_q$, as for an element $x \in \mathbb{F}_{q^2}$, its norm $x^{q+1} \in \mathbb{F}_q$. Therefore, $P(C)$ is a constacyclic code with shift constant $\tilde{\beta}^n$, and its generator polynomial has roots $\{\tilde{\beta}\alpha^i\colon i, j \in \mathcal{Z}'\}$, where $\mathcal{Z}'$ is defined in (11). ∎

For the MDS codes from Theorem 5, the defining set $\mathcal{Z}$ consists of $d-1$ consecutive numbers. Based on the computational results in Section VI below, we have the following conjecture for the corresponding puncture code:

Conjecture 11: Let $C^* = [q^2 + 1, q^2 = 1 - d, d]_{q^2}$ be an $\mathbb{F}_{q^2}$-linear (consta)cyclic MDS code.

Then the corresponding puncture code $P(C)$ has parameters $PC = [q^2 + 1, q^2 + 1 - (d-1)^2, d']_q$ where

$$d' = \begin{cases} 2(d-1) & \text{for } 1 < d \leq q/2 + 1 \\ (q+1)(d-1-\lfloor q/2 \rfloor)) & \text{for } q/2 + 1 < d \leq q, \ q \text{ odd} \\ q(d - \lfloor q/2 \rfloor) & \text{for } q/2 + 1 < d \leq q, \ q \text{ even} \\ q^2 + 1 & \text{for } d = q + 1 \end{cases} \tag{15}$$

Using the Hartmann-Tzeng bound for (consta)cyclic codes (see, e.g., [16]), we get the lower bound $d' \geq 2(d-1)$. Using the Roos bound, we get a better lower bound for $d > q/2 + 1$, but in general the conjectured minimum distance d' is even larger.

V. RESULTS

A. QMDS Codes of Minimum Distance Two

In [12, Table II], the existence of QMDS codes $\mathcal{C} = [\![n, n-2, 2]\!]_q$ is stated for the case that the length n is divisible by the characteristic p of the field $\mathbb{F}_q$, i.e., $q = p^m$, p prime. In that case, the classical repetition code $C = [n, 1, n]_q$ is contained in its Euclidian dual $C^\perp = [n, n-1, 2]_q$, and the CSS construction yields the corresponding quantum code. More generally, consider the classical repetition code $C = [n, 1, n]_{q^2}$ over the field $\mathbb{F}_{q^2}$. Then the puncture code is the MDS code $P(C) = [n, n-1, 2]_q$. When $q > 2$, $P(C)$ contains words of all weights $2 \leq w \leq n$ (see [5]). By [17, Theorem 14], the existence of codes $[\![n, n-2, 2]\!]_{q_1}$ and $[\![n, n-2, 2]\!]_{q_2}$ implies the existence of a code $[\![n, n-2, 2]\!]_{q_1 q_2}$. In summary, we have

Theorem 12: Let $q > 1$ be an arbitrary integer, not necessarily a prime power. Quantum MDS codes $\mathcal{C} = [\![n, n-2, 2]\!]_q$ exist for all even length n, and for all length $n \geq 2$ when the dimension q of the quantum systems is an odd integer or is divisible by 4.

B. QMDS Codes of Length $q^2 + 1$

Theorem 13: Our construction yields QMDS codes with parameters $\mathcal{C} = [\![q^2+1, q^2+3-2d, d]\!]_q$ for all $1 \leq d \leq q+1$ when q is odd, or when q is even and d is odd.

Proof: When the minimum distance is $d = q+1$, the puncture code has q^2 consecutive roots and is hence a trivial MDS code $P(C) = [q^2+1, 1, q^2+1]_q$. In particular, $P(C)$ contains a word of weight q^2+1, and hence a QMDS code $\mathcal{C} = [\![q^2+1, q^2+1-2q, q+1]\!]_q$ exists. Note that the MDS codes of even dimension from Theorem 5 form a chain of nested codes, and likewise the codes of odd dimension. Using Lemma 9, it follows that $P(C)$ contains a word of weight q^2+1 whenever the minimum distance d of C^* has the same parity as $q+1$.

When q is odd and $d = q$, the classical MDS code is a constacyclic code with generator polynomial $g_3(z)$ given by (7). The corresponding puncture code $P(C)$ is a constacyclic code over $\mathbb{F}_q$ of length q^2+1 with defining set $\mathcal{Z}' = \{i + qj : i, j = -t+1, \ldots, t\}$, $t = (q-1)/2$. The generator polynomial of $P(C)$ divides the polynomial $f(z) = z^{q^2+1} - \gamma$, where $\gamma \in \mathbb{F}_q$ is a primitive element of $\mathbb{F}_q$. As q is odd, γ has two square roots $\pm\sqrt{\gamma} \in \mathbb{F}_{q^2}$. Moreover, for any $x \in \mathbb{F}_{q^2}$, $x^{q^2+1} = x^2$ which shows that $\pm\sqrt{\gamma}$ are roots of $f(z)$ and hence the polynomial $f(z)$ is divisible by $z^2 - \gamma$. The corresponding defining set is $\{(q^2+1)/2, (q^4+q^2)/2+1\}$ which is disjoint from $\mathcal{Z}'$. Hence $\pm\gamma$ are not among the roots of $g_3(z)$ and $g_3(z)$ divides the polynomial $g(z) = f(z)/(z^2-\gamma)$. The constacyclic code generated by $g(z)$ is the sum of two trivial MDS codes $[(q^2+1)/2, 1, (q^2+1)/2]_q$. In particular it contains a word of weight q^2+1. Using Lemma 9, it follows that $P(C)$ contains a word of weight q^2+1 for q odd and when the minimum distance d of C^* is odd as well. ∎

Note that when both q and d are even, the theorem does not hold in general. For example, our construction does not yield a QMDS code $[\![17, 11, 4]\!]_4$. However, this might be the only exception as for $q = 2^m$, $m = 3, 4, 5, 6, 7$, our construction provides codes $[\![4^m+1, 4^m+3-2^{m+1}, 2^m]\!]_{2^m}$. To show this, we find a cyclic subcode of $P(C)$ of dimension 4 which contains a word of weight 4^m+1. This also implies the existence of QMDS codes of length 2^m+1 for all distances $d \leq 2^m+1$.

C. QMDS Codes of Length $q^2 + 2$

For $q^2 = 2^{2m}$, there exist classical MDS codes with parameters $C^* = [2^{2m}+2, 2^{2m}-1, 4]_{2^{2m}}$ (see, e.g., [15, Ch. 11, §5, Theorem 10]). A parity check matrix for C^* is

$$H = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\ \alpha^0 & \alpha^1 & \alpha^2 & \ldots & \alpha^{q^2-2} & 0 & 1 & 0 \\ \alpha^0 & \alpha^2 & \alpha^4 & \ldots & \alpha^{2(q^2-2)} & 0 & 0 & 1 \end{pmatrix}, \quad (16)$$

where α denotes a primitive element of $\mathbb{F}_{q^2}$. The code C is not self-orthogonal, as, for example, the Hermitian inner product of the second row of H with itself is non-zero.

Theorem 14: For $q = 2^m$, there exist QMDS codes with parameters $\mathcal{C} = [\![4^m+2, 4^m-4, 4]\!]_{2^m}$.

Proof: A parity check matrix $H_{P(C)}$ of the puncture code $P(C)$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\ 1 & \alpha^{q+1} & \alpha^{2(q+1)} & \ldots & \alpha^{(q^2-2)(q+1)} & 0 & 1 & 0 \\ 1 & \alpha^{2(q+1)} & \alpha^{4(q+1)} & \ldots & \alpha^{2(q^2-2)(q+1)} & 0 & 0 & 1 \\ 1 & \alpha^1 & \alpha^2 & \ldots & \alpha^{q^2-2} & 0 & 0 & 0 \\ 1 & \alpha^2 & \alpha^4 & \ldots & \alpha^{2(q^2-2)} & 0 & 0 & 0 \\ 1 & \alpha^q & \alpha^{2q} & \ldots & \alpha^{q(q^2-2)} & 0 & 0 & 0 \\ 1 & \alpha^{q+2} & \alpha^{2(q+2)} & \ldots & \alpha^{(q+2)(q^2-2)} & 0 & 0 & 0 \\ 1 & \alpha^{2q} & \alpha^{4q} & \ldots & \alpha^{2q(q^2-2)} & 0 & 0 & 0 \\ 1 & \alpha^{2q+1} & \alpha^{4q+2} & \ldots & \alpha^{(2q+1)(q^2-2)} & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

where α denotes a primitive element of $\mathbb{F}_{q^2}$. The first three rows of $H_{P(C)}$ are given by the componentwise norm of the matrix H defined in (16). In particular, the entries of these three rows lie in $\mathbb{F}_q$. Consider the matrix

$$\begin{aligned} G_1 &= \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\ 1 & \alpha^{-(q+1)} & \alpha^{-2(q+1)} & \ldots & \alpha^{-(q^2-2)(q+1)} & 0 & 1 & 0 \\ 1 & \alpha^{-2(q+1)} & \alpha^{-4(q+1)} & \ldots & \alpha^{-2(q^2-2)(q+1)} & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\ 1 & \beta & \beta^2 & \ldots & \beta^{q^2-2} & 0 & 1 & 0 \\ 1 & \beta^2 & \beta^4 & \ldots & \beta^{2(q^2-2)} & 0 & 0 & 1 \end{pmatrix}, \quad (18) \end{aligned}$$

where $\beta = \alpha^{-(q+1)}$ is a primitive element of $\mathbb{F}_q$. The matrix G_1 is obtained by taking the inverse of the non-zero entries in the first three rows of $H_{P(C)}$ given in (17). Each row of G_1 is orthogonal to all rows of $H_{P(C)}$, i.e., the span of G_1 is contained in $P(C)$. Let $f(x) = x^2 + \gamma_1 x + \gamma_0$ be an irreducible polynomial over $\mathbb{F}_q$. Then in particular $\gamma_1 \neq 0 \neq \gamma_0$. Furthermore, every coordinate of the codeword $\boldsymbol{c} = (\gamma_0, \gamma_1, 1)G_1$ is non-zero, as the last three coordinates of $\boldsymbol{c}$ just equal $(\gamma_0, \gamma_1, 1)$, and the other coordinates correspond to the evaluation of the irreducible polynomial $f(x)$ at some power of β. Therefore, $P(C)$ contains the codeword $\boldsymbol{c}$ of weight q^2+2. ∎

The computational results show that for $q = 2^m$, $m = 3, \ldots, 7$ the puncture code $P(C)$ of the MDS code with parity check matrix H given in (16) does not only contain a word of weight 4^m+2, but words of all weights $6 \leq w \leq 4^m+2$. This suggests the following:

Conjecture 15: For $q = 2^m$, $q \neq 4$, there exist QMDS codes with parameters $[\![n, n-6, 4]\!]_{2^m}$ for all $6 \leq n \leq 4^m+2 = q^2+2$.

VI. COMPUTATIONAL RESULTS

A. Qubit Codes

For $q = 2$, we only have the QMDS codes $[\![5, 1, 3]\!]_2$, $[\![6, 0, 4]\!]_2$, as well as the codes of even length $[\![2m, 2m-2, 2]\!]_2$ and the trivial codes $[\![n, n, 1]\!]_2$.

B. Qutrit Codes

For $q = 3$, in addition to the trivial codes with distance $d = 1, 2$, our construction yields QMDS codes $[\![n, n-4, 3]\!]_3$ for $n = 4, \ldots, 10$ as well as the code $[\![10, 4, 4]\!]_3$.

What is more, from Hermitian self-dual MDS codes $C = [6, 3, 4]_9$ and $C = [10, 5, 6]_9$ in [9], we get QMDS codes $\mathcal{C} = [\![6, 0, 4]\!]_3$ and $\mathcal{C} = [\![10, 0, 6]\!]_3$, respectively. By Corollary 4, from $\mathcal{C}$ we obtain QMDS codes $[\![9, 1, 5]\!]_3$ and $[\![8, 2, 4]\!]_3$

C. Ququad Codes

For $q = 4$, in addition to the trivial codes with distance $d = 1, 2$, our construction yields QMDS codes $[\![n, n-4, 3]\!]_4$ for $n = 4, \ldots, 17$, codes $[\![n, n-6, 4]\!]_4$ for even $n = 8, 10, \ldots, 18$, as well the code $[\![17, 9, 5]\!]_4$. Codes with distance $d = 4$ and odd length cannot be directly obtained, as the puncture code is even in this case. However, by direct search we found codes with parameters $[\![6, 0, 4]\!]_4$, $[\![9, 3, 4]\!]_4$, and $[\![11, 5, 4]\!]_4$. From a Hermitian self-dual MDS code of length 10 given in [8], we obtain QMDS codes $[\![10, 0, 6]\!]_4$ and $[\![9, 1, 5]\!]_4$.

D. Ququint Codes

For $q = 5$, in addition to the trivial codes with distance $d = 1, 2$, our construction yields QMDS codes $[\![n, n-4, 3]\!]_5$ for $n = 4, \ldots, 26$, codes $[\![n, n-6, 4]\!]_5$ for $n = 6$ and $n = 8, \ldots, 18$, codes $[\![n, n-8, 5]\!]_5$ for $n = 12, \ldots, 26$, and the code $[\![26, 16, 6]\!]_5$. For $d = 4$, the puncture code does not contain a word of weight 7. Hermitian self-dual MDS codes from [8] yield QMDS codes $[\![8, 0, 5]\!]_5$ and $[\![10, 0, 6]\!]_5$. Applying Corollary 4 to these codes, we obtain the missing QMDS code $[\![7, 1, 4]\!]_5$ and in addition a code $[\![9, 1, 5]\!]_5$. Moreover, a QMDS code $[\![10, 2, 5]\!]_5$ was found by randomized search.

E. Qusept Codes

For $q = 7$, we obtain all QMDS codes $[\![n, n+2-2d, d]\!]_7$ for all $2d - 2 \leq n \leq 50$, $2 \leq d \leq 4$. For $d = 5$, the puncture codes does not contain words of weight $w = 9, 10, 11$, and for $d = 6$, $P(C)$ contains words in the range $16, \ldots, 50$, with the exception of $w = 17$. For $d = 7$, $P(C)$ contains words in the range $16, \ldots, 50$, with the exception of $w = 26, 27, 29$. For $d = 8$ we have the code $[\![50, 36, 8]\!]_7$ from our construction. From Hermitian self-dual MDS codes in [8], we obtain QMDS codes $[\![10, 0, 6]\!]_7$, $[\![12, 0, 7]\!]_7$, and $[\![14, 0, 8]\!]_7$. Using Corollary 4, we also get the missing codes $[\![9, 1, 5]\!]_7$, $[\![10, 2, 5]\!]_7$, and $[\![11, 3, 5]\!]_7$ with distance 5, as well as the additional shorter codes $[\![11, 1, 6]\!]_7$, $[\![13, 1, 7]\!]_7$, and $[\![12, 2, 6]\!]_7$. We have not yet found a code $[\![17, 7, 6]\!]_7$ or the QMDS codes of length $n = 26, 27, 29$ and distance 7.

F. Quoct Codes

For $q = 8$, we obtain almost all of the QMDS codes implied by Conjecture 11. For $d = 8$, the puncture code $P(C)$ does not contain words of weight $33, 34, 35, 37, 39$. For $d = 7$, we have all weights in the range $24, \ldots, 65$. For $d = 6$, the puncture code $P(C)$ contains words of weights $16, 18, \ldots, 65$, but random sampling did not reveal a word of weight 17. The corresponding code $[\![17, 5, 6]\!]$ has not yet been found either. For $d = 5$, random sampling of the puncture code $P(C)$ did not reveal words of weight $w = 9, 10, 11$, but the corresponding

codes $[\![9, 1, 5]\!]_8$, $[\![10, 2, 5]\!]_8$, and $[\![11, 3, 5]\!]_8$ can be derived from Hermitian self-dual codes $[10, 5, 6]_{64}$, $[12, 6, 7]_{64}$, and $[14, 7, 8]_{64}$.

G. Qudit Codes with $9 \leq q \leq 32$, $q = 64$

While we have complete information about the weight spectra of the puncture codes for $q \leq 8$, the information about the weights in $P(C)$ for $q \geq 9$ is mainly based on sampling codewords. In Fig. 2 we plot the minimum distance (y-axis) against the length of QMDS codes obtained by our construction for $q = 8, \ldots, 17$. We also plot the conjectured lower bounds on the minimum weight of $P(C)$. The squares indicate QMDS codes that were found by different methods. Note that the steeper bound corresponds to the quantum Singleton bound for $k = 0$. With increasing alphabet size q, and hence increasing length $q^2 + 1$, it becomes more difficult to find codewords of low weight in $P(C)$. For $q = 16$, we use subfield-subcodes over $\mathbb{F}_4$ and $\mathbb{F}_2$. Fig. 1 and Fig. 3 show the corresponding data for $q = 3, 4, 5, 7$ and $q = 19, \ldots, 32, 64$, respectively. Again, for $q = 5^2, 3^3, 2^5, 2^6$ we use subfield-subcodes of $P(C)$ as well.
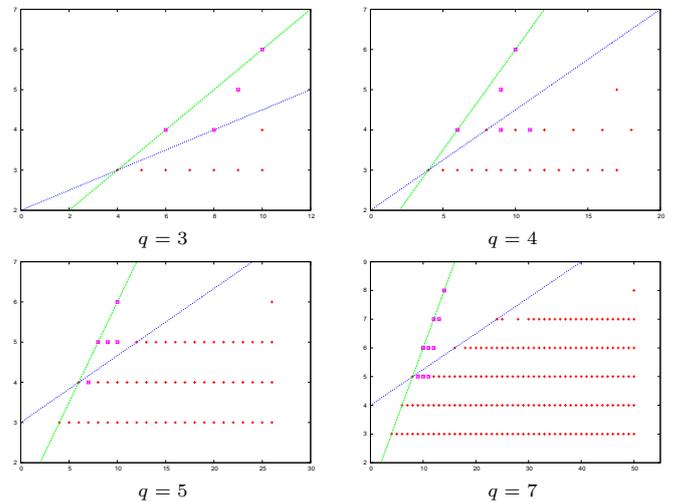

Fig. 1. Minimum distance and length of QMDS codes for dimensions $3 \leq q \leq 7$.

VII. Conclusion

In analogy to the conjecture for classical MDS codes, it has been conjectured that the length of non-trivial quantum MDS codes is bounded by $q^2 + 1$, or $q^2 + 2$ for very specific values [12]. Our results suggest that we can indeed find MQDS codes for all lengths up to this bound. Somewhat surprisingly, it seems that with increasing minimum distance, it might become more difficult to find QMDS codes below a certain length. Moreover, our construction is restricted to QMDS codes of minimum distance $q + 1$. Nonetheless, we found QMDS codes $[\![10, 0, 6]\!]_3$, $[\![9, 1, 5]\!]_3$, and $[\![10, 0, 6]\!]_4$ with $d > q + 1$.

We finally note that we can apply the technique of the puncture code also to classical MDS codes of length $q^2 - 1$ or q^2 based on (extended) Reed-Solomon codes. Obviously, this does not yield QMDS codes of length $q^2 + 1$, but in terms of the weight spectra of the codes $P(C)$, and hence the achievable lengths of QMDS codes, we get essentially the same results.
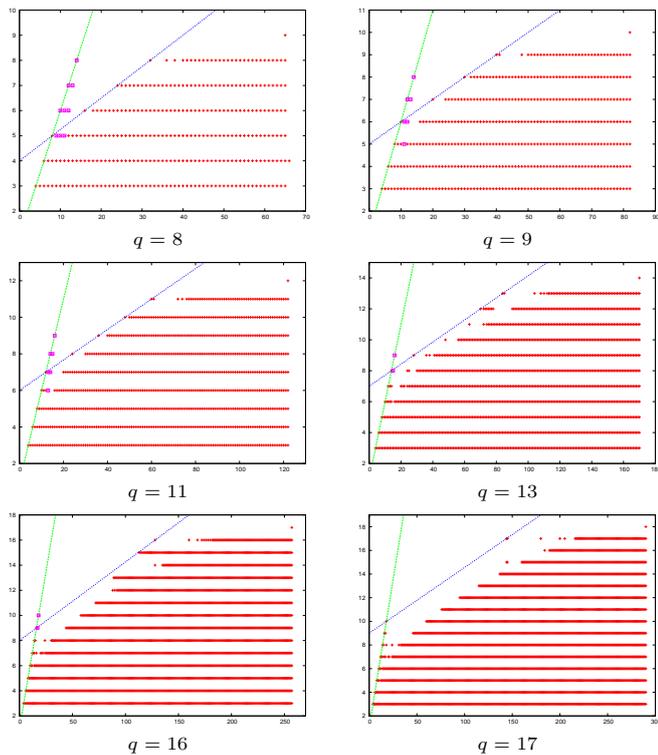
Fig. 2.   Minimum distance and length of QMDS codes for dimensions $8 \leq q \leq 17$.

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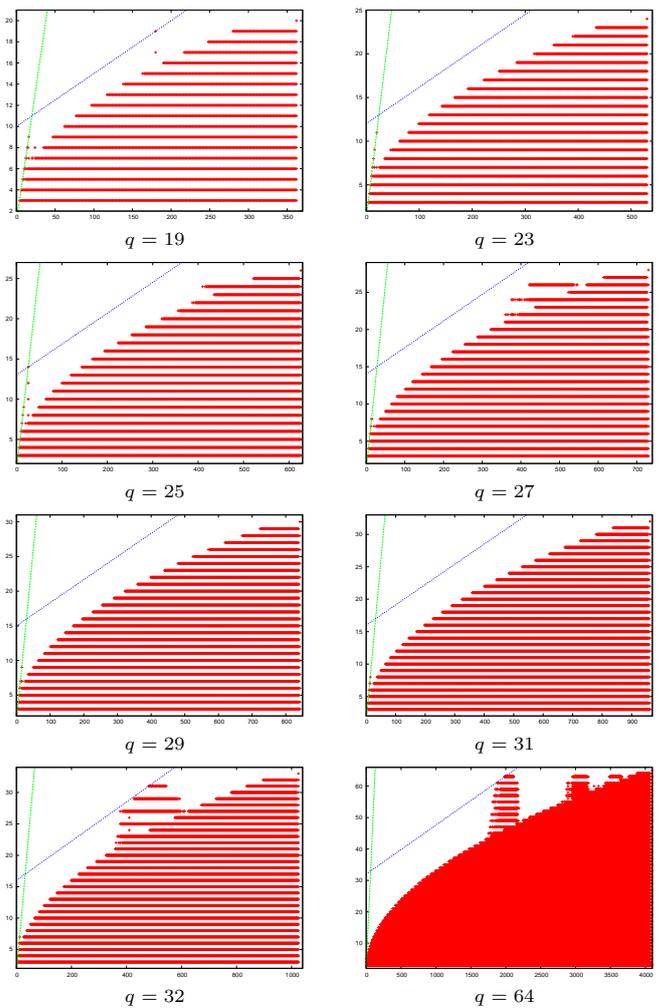
Fig. 3.   Minimum distance and length of QMDS codes for dimensions $q \leq 19 \leq 32$ and $q = 64$.

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