SEQUENTIAL AUCTIONS WITH RANDOMLY ARRIVING BUYERS

MAHER SAID†

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ABSTRACT: We analyze a dynamic market in which buyers compete in a sequence of auctions for common value or differentiated goods. New buyers and objects may arrive at random times. Since objects are imperfect substitutes, buyers' private values are not persistent. Instead, buyers receive new signals or draw new values in each period.

We consider the use of second-price auctions for selling these objects. In equilibrium, buyers do not bid their true expected values. Instead, they shade their bids down by their continuation value, which is the option value of participating in future auctions. We show that this option value depends not only on the number of buyers currently present on the market, but also on anticipated market dynamics. In particular, the option value also depends on the arrival rates on both sides of the market. We extend our results to the setting in which objects come from different distributions and market conditions evolve, with some persistence, as in a "buyer's market" or a "seller's market."

KEYWORDS: Dynamic markets, Sequential auctions, Endogenous options, Random arrivals, Stochastic equivalence.

JEL CLASSIFICATION: C73, D44, D83.

†MICROSOFT RESEARCH NEW ENGLAND AND OLIN BUSINESS SCHOOL, WASHINGTON UNIVERSITY IN ST. LOUIS, MSAID@MICROSOFT.COM

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1. Introduction

In this paper, we consider a model of dynamic markets in which buyers and sellers arrive randomly to the market. We examine the equilibrium behavior of market participants in response to these arrivals, as well as to changes in market conditions. In particular, we investigate the role of current and anticipated future conditions in determining endogenous outside options in this setting. We characterize the influence of these outside options on equilibrium price determination, and study the manner in which equilibrium payoffs and behavior are affected by both current conditions and future dynamics.

Consider a buyer participating in a pure common values auction; for instance, a firm bidding for oil drilling rights on a tract of land. The firm has conducted a geological survey and has an estimate of the value of the tract, and will base its bid on that estimate (while accounting for the winner's curse). However, the firm's bid will also depend on the possibility of participating in future auctions for the rights to other tracts. Moreover, these future tracts involve new surveys and new estimates, and hence the firm's value for each tract differs—some tracts may appear to be especially valuable to the firm, while others may appear to be less so. Moreover, different firms take different samples, the nature of this heterogeneity is unpredictable across firms: if firm's survey capabilities are similar, a particular firm is just as likely to be overly optimistic about a given tract as it is to be overly pessimistic. What is more, some competitors may either join or leave the market in the interim. Thus, the frequency of future auctions, as well as the expected competition in those auctions, naturally affects the trade-offs faced by the firm in the present. Rational bidding behavior by the firm must therefore account for these trade-offs as part of the bid determination process.

Alternately, consider an individual who wants to purchase a new home. After surveying the houses available on the market, a potential buyer will determine which house best matches her individual preferences as well as her budgetary constraints, and will make an offer on that home. Obviously, the amount of her offer will depend on the physical characteristics of the home: the size of the house, the neighborhood or school district in which it lies, the potential for resale in the future, and so on. Note that this heterogeneity is evaluated differently by different buyers. In addition, however, market conditions will play a large role. In a "seller's market" in which demand for housing is large relative to supply, prices will be higher. Abstracting away from immediate housing needs, expectations about future market conditions and the dynamics governing them will also play an important role. If high demand is expected to be only a short-lived phenomenon, prices will be attenuated somewhat, especially if buyers are patient. Similarly, if a large number of homes are expected to come onto the market in the near future, prices will be further depressed. Thus, the offer of a potential homebuyer will depend on both the current competitiveness of the market as well as the anticipated characteristics of the market in the near future.

The present work abstracts away from many of the "fine" details of these various markets, focusing on what we view as the essential features. In particular, we examine the situation in which buyers are confronted with an infinite sequence of auctions for heterogeneous but stochastically equivalent objects that arrive at random times. Moreover, new buyers probabilistically arrive on the market in each period. Thus, in any given auction, buyers are presented with the outside option of participating in a future auction for an imperfectly substituted "equivalent" object, but with a potentially different number of competitors. We provide a precise characterization of this option value, and explore the manner in which it varies with the number of buyers currently present on the market, as well as with expectations of future market conditions.

Essentially, losing in an auction today yields the opportunity to participate in another auction in the future; however, the potential for entry by additional buyers and the random arrival times of auctions implies, in contrast to much of the literature on sequential auctions, that the competitive environment in the future may differ significantly from the present one. Thus, the expected payoff of a buyer currently present on the market is directly linked to her expected payoffs with a different number of competitors present in future periods. This leads to a difference equation characterizing the "outside option" available to each buyer, which is endogenously determined in equilibrium. We show that this outside option is, in fact, an appropriately discounted sum of expected payoffs from participating in each of the infinite sequence of auctions with differing numbers of participants, where the weight on each auction is a combination of pure time discounting and the likelihood of market dynamics leading to the corresponding state. Moreover, optimal bidding behavior accounts for the option value in a straightforward and separable manner. Finally, this result is robust to various assumptions regarding the evolution of valuations, as well as to changes in the trading institution employed.

The present work is related to, and complements, several strands of the literature on dynamic markets and bargaining. Wolinsky (1988) considers the steady state of a dynamic market where bidders are randomly matched to sellers. Buyers view all objects as imperfect substitutes and engage in a first-price sealed-bid auction. As in the present work, buyers may wish to delay trade so as to "find" an object that is more valuable to them. On the other hand, De Fraja and Sákovics (2001) examine a market where buyers have persistent private information, and so buyers' only incentive to delay trade is to obtain a better price in the future. Taylor (1995) examines bargaining power and price formation as it relates to the number of traders on each side of a market. His model assumes that agents are homogeneous—all buyers have the same commonly known value for purchasing an object—and that trade is conducted via posted prices. As in Coles and Muthoo (1998) and Coles and Smith (1998), our model enriches his setting by allowing for heterogeneous buyers and objects. Instead of a combination of ultimatum bargaining and Betrand competition, however, we model the surplus-division process by explicitly employing an auction mechanism for price determination. In this sense, our model is also related to Satterthwaite and Shneyerov (2007). These authors consider a world in which a continuum of both buyers and sellers enter in every period, following time-invariant strategies in a steady-state equilibrium. The present work, on the other hand, is concerned with the behavior of agents in a dynamic environment with constantly changing conditions, and so a steady-state analysis runs counter to its goals.

Inderst (2008) considers a bargaining model in which a seller is randomly visited by heterogeneous buyers. If the seller is currently engaged in bargaining with one agent when another arrives, she may choose to switch from bargaining with one buyer to bargaining with the other. However, this switch is permanent, implying that the arrival of a new buyer either "restarts" the game or is completely irrelevant. Fuchs and Skrzypacz (2010) take a different approach: they consider an incomplete information bargaining problem between a buyer and a seller, and allow for the possibility of the arrival of various "events." These events end the game and yield an exogenously determined expected payoff to each agent. The suggested interpretation is that these events may be viewed as triggers for some sort of multi-lateral mechanism involving new entrants (a second-price auction, for example) for which the expected payoffs are a reduced-form representation. Thus, while both works are primarily concerned with characterizing the endogenous option value that results from the potential arrival of additional participants to the market, they do this in a framework of bilateral bargaining which does not capture the dynamic nature of direct competition between several current and future potential market participants.[1]

The present work is also closely linked to elements of the literature on sequential auctions. Engelbrecht-Wiggans (1994) studies sequential auctions in which a fixed number of perfectly patient buyers with single-unit demand compete in a finite sequence of second-price auctions for stochastically equivalent objects. His model, however, does not allow for several features of the present work; in particular, it does not allow for the entry of new buyers or consider the role of market dynamics in price determination. Zeithammer (2009) considers a similar environment, restricted to two periods (and two objects), and studies the efficiency properties of sequential auctions when buyers initially know their value for each of the two objects. Budish (2008), in a closely related model, examines the effects of information revelation about future objects on efficiency. Said (2010) looks at a setting with entry dynamics similar to the present work, but makes the complementary (and opposite) assumption of independent private values that are *fixed* across time, focusing on the design of optimal mechanisms. As in the present work, outside options are endogenously determined; however, the presence of persistent private values introduces an element of learning that is not present herein.

Sailer (2006) and Zeithammer (2006) both conduct empirical examinations of eBay auctions while taking into consideration the sequential nature of that market. Although the latter is differentiated by an assumption that buyers are able to observe their valuation for some upcoming objects, both authors assume a fixed number of competing buyers in each period, and therefore are unable to account for fluctuations in market conditions and competitiveness. Essentially, they assume away the existence of variation in market conditions. Thus, they are closely related to the special case of our model in which an auction occurs in every period and winning bidders are always immediately replaced by exactly one new buyer, ensuring that stationary market conditions. Jofre-Bonet and Pesendorfer (2003) assume stochastically equivalent objects and allow for the entry of new potential buyers, but these agents are short-lived bidders who do not take the future into account. Finally, Backus and Lewis (2010) and Ingster (2009) are more recent examples of the empirical work on eBay that allow for market fluctuations and dynamics; instead of assuming stationarity in market conditions, however, they make relatively strong anonymity or large-market assumptions in order to characterize bidder behavior.[2]

[1] Deb (2009) examines a related model where the arrival of an "event" is unobservable to the seller and, as in the present work, continuation (or "option") values are endogenously determined.

[2] Nekipelov (2008) also models random entry in eBay auctions. However, his model is concerned with the role of new entrants *within* a *single* auction, and hence is best viewed as complementary to this line of research.

The paper is organized as follows. Section 2 presents our model, and Section 3 solves for the symmetric Markov equilibrium. Section 4 discusses some comparative statics results. Section 5 demonstrates the robustness of the model in a setting where market characteristics may vary from one period to the next. Finally, Section 6 concludes, suggesting some avenues for further research.

2. THE MODEL

We consider the continuous-time limit of an infinite-horizon discrete-time market model; periods of length Δ are indexed by $t \in \mathbb{N}$. There is a finite number $n_t \geq 2$ of risk-neutral buyers with single-unit demand on the market in any given period t. Since the objects under consideration are differentiated and hence imperfectly substitutable, buyers have different values for different goods. Moreover, since buyers are heterogeneous, their evaluation of these distinct goods differ from one another—in the terminology of Engelbrecht-Wiggans (1994), objects are stochastically equivalent.[3] This also arises if objects are common-value goods and each buyer receive a new unbiased signal about each object.

Thus, each buyer $i \in \{1, \ldots, n_t\}$ has a private valuation v_i^t for the object available at time t, where v_i^t is independently (across buyers i and periods t) drawn from the distribution F on $\mathbb{R}_+$. We assume that F has finite variance and a continuous density function f. In the case of a sequence of pure common values auctions, as in the introductory example, v_i^t may be thought of as buyer i's expected value for the object at hand (given her signal) when she is pivotal—her *ex interim* willingness to pay. Finally, buyers discount the future exponentially with discount factor $\delta = e^{-r\Delta}$, where the discount rate is $r > 0$.

In each period, there is at most one seller present. The arrival of sellers is stochastic; in particular, there is some exogenously fixed probability $p = \lambda\Delta, \lambda > 0$ that a new seller arrives on the market in each period. Similarly, additional buyers may arrive on the market in each period. For simplicity, we will assume that at most one buyer arrives at a time, and that this arrival occurs with some exogenously given probability $q = \rho\Delta, \rho > 0$. Note, however, that we assume that this arrival occurs immediately and with probability 1 if there is only a single buyer remaining on the market from previous periods—there are always at least two buyers present. In the event that a seller has arrived on the market, each buyer i observes both v_i^t and the number of competing buyers present. The seller then conducts a second-price auction to allocate their object.

We assume that sellers are nonstrategic—they are unable to set a reserve price and cannot remain on the market for more than one period. Conversely, buyers must participate in each auction that takes place when they are present on the market, but may submit bids less than their true values.[4] There are two main advantages to assuming that reserve prices are not used. First, we are able to avoid the issues of pooling (partial or otherwise) that can occur at the reserve price when there are allocational externalities—see, for instance, Haile (2000) and Jehiel and Moldovanu (2000). Second, equilibrium in the presence of reserve prices is characterized by a *nonlinear* second-order difference equation. Solving such a difference equation numerically, let alone analytically, is

[3] This assumption is not uncommon in the literature on dynamic markets, appearing in (among others) Wolinsky (1988), Coles and Muthoo (1998), Coles and Smith (1998), Jofre-Bonet and Pesendorfer (2003), Sailer (2006), and Budish (2008).

[4] It is easy to see that, in equilibrium, bids will always be positive if the support of the value distribution is sufficiently high (given a particular discount factor). As the discount factor decreases, this bound approaches zero.

typically an exercise in futility. And, computational issues aside, the lack of a closed-form solution precludes many avenues of additional investigation, including (for instance) standard comparative statics exercises. This is in sharp contrast to, for instance, Coles and Muthoo (1998), who arrive at a *linear* second-order (homogenous) difference equation in their model, in part due to their assumptions of binary values and price-setting via Bertrand competition.

As is standard, we write $Y_k^{(n)}$ to denote the k-th highest of n independent draws from F, with $G_k^{(n)}$ and $g_k^{(n)}$ denoting the corresponding distribution and density functions, respectively, of this random variable. In addition, we will define, for all $n \in \mathbb{N}$,

$$\widehat{Y}(n) := \mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_1^{(n-1)}\right].$$

This is the expected difference between the highest of n and $n-1$ independent draws from F, where, by convention, we let $\mathbb{E}\left[Y_1^{(0)}\right] = 0$. It is useful to note that $\widehat{Y}(n)$ is decreasing in n.

3. Symmetric Markov Equilibrium

Let $V(v_i^t, n)$ denote the expected payoff to a bidder when her valuation is v_i^t and she is one of n bidders present on the market. Recall that a seller must be currently present on the market for buyers to be aware of their valuations. Furthermore, let $W(n)$ denote the expected value to a buyer when she is one of n buyers present on the market at the beginning of a period, *before* the realization of the buyer and seller arrival processes. At the beginning of a period when there are $n \geq 2$ buyers present, there are four possible outcomes: with probability pq, both a buyer and a seller may arrive, leading to an auction with $n+1$ participants; with probability $p(1-q)$ only a seller arrives, yielding an auction with n participants; with probability $(1-p)q$ only a buyer may arrive, leading to the next period starting with $n+1$ participants; or, with the remaining probability $(1-p)(1-q)$, neither a buyer nor a seller may arrive, leading to the next period being identical to the current one. Thus, for all $n \geq 2$,

$$\begin{aligned} W(n) := {} & pq\mathbb{E}[V(v_i^t, n+1)] + p(1-q)\mathbb{E}[V(v_i^t, n)] \\ & + (1-p)q\delta W(n+1) + (1-p)(1-q)\delta W(n). \end{aligned} \tag{1}$$

Let us now consider the problem facing buyer i when there are $n \geq 2$ buyers on the market and an object is currently available (and, hence, an auction is "about" to occur). This buyer, with valuation v_i^t must choose her bid b_i^t. If she wins the auction, she receives a payoff of v_i^t less the second-highest bid. On the other hand, if she loses, she remains on the market as one of $n-1$ buyers tomorrow, yielding her a payoff of $\delta W(n-1)$. Therefore,

$$V(v_i^t, n) = \max_{b_i^t} \left\{ \begin{array}{c} \Pr\left(b_i^t > \max_{j \neq i}\{b_j^t\}\right) \mathbb{E}\left[v_i^t - \max_{j \neq i}\{b_j^t\} | b_i^t > \max_{j \neq i}\{b_j^t\}\right] \\ + \Pr\left(b_i^t < \max_{j \neq i}\{b_j^t\}\right) \delta W(n-1) \end{array} \right\}.$$

We may use this expression in order to determine equilibrium bid functions, as demonstrated in the following result. Note that we focus on the unique symmetric Markov equilibrium of this dynamic game. Other equilibria certainly exist; however, as all other equilibria are ruled out by interim dominance arguments, it is natural to focus on the symmetric Markov equilibrium.

LEMMA 1 (Equilibrium bids).
In equilibrium, a buyer with value v_i^t who is one of $n \geq 2$ buyers on the market bids $b_i^t = b^*(v_i^t, n)$, where

$$b^*(v_i^t, n) := v_i^t - \delta W(n-1). \tag{2}$$

PROOF. Note that, since

$$\Pr(b_i^t < \max_{j \neq i}\{b_j^t\}) = 1 - \Pr(b_i^t > \max_{j \neq i}\{b_j^t\}),$$

we may rewrite $V(v_i^t, n)$ as

$$\max_{b_i^t} \left\{ \begin{array}{c} \Pr(b_i^t > \max_{j \neq i}\{b_j^t\})\mathbb{E}\left[v_i^t - \delta W(n-1) - \max_{j \neq i}\{b_j^t\} | b_i^t > \max_{j \neq i}\{b_j^t\}\right] \\ +\delta W(n-1) \end{array} \right\}.$$

Since the trailing $\delta W(n-1)$ in the above expression is merely an additive constant, the maximization problem above is corresponds to that of a second-price auction with n bidders in which each bidder i's valuation is given by $v_i^t - \delta W(n-1)$. The standard dominance argument for second-price auctions then implies that $b^*(v_i^t, n) = v_i^t - \delta W(n-1)$. □

Given this bidding strategy and the fact that continuation payoffs do not differ across buyers, the probability of i winning the auction in period t is simply $\Pr(v_i^t > \max_{j \neq i}\{v_j^t\})$, and the surplus gained in this case becomes $v_i^t - \max_{j \neq i}\{v_j^t\}$. Therefore, we may rewrite V as

$$\begin{aligned} V(v_i^t, n) &= \Pr(v_i^t > \max_{j \neq i}\{v_j^t\})\mathbb{E}\left[v_i^t - \max_{j \neq i}\{v_j^t\} | v_i^t > \max_{j \neq i}\{v_j^t\}\right] + \delta W(n-1) \\ &= \Pr\left(Y_1^{(n-1)} < v_i^t\right)\mathbb{E}\left[v_i^t - Y_1^{(n-1)} | Y_1^{(n-1)} < v_i^t\right] + \delta W(n-1) \\ &= G_1^{(n-1)}(v_i^t)\left(v_i^t - \mathbb{E}\left[Y_2^{(n)} | Y_1^{(n)} = v_i^t\right]\right) + \delta W(n-1), \end{aligned} \tag{3}$$

where the equivalence between the second and third lines relies on the properties of the highest and second-highest order statistics.[5]

Note that, ex ante, any one of the $n \geq 2$ buyers present on the market in any period is equally likely to have the highest value amongst her competitors (and hence win the object). We may use this fact, along with the result above, to show that the ex ante expected utility of a buyer when there is an object available for sale is simply the sum of her probability of winning the object multiplied by her expected payoff, conditional on winning, and the discounted option value of losing the object and remaining on the market in the next period. Formally, we are able to prove the following result.

LEMMA 2 (Expected auction payoffs).
The expected payoff to a bidder from an auction with $n \geq 2$ participants is

$$\mathbb{E}[V(v_i^t, n)] = \widehat{Y}(n) + \delta W(n-1). \tag{4}$$

[5]See, for example, David and Nagaraja (2003, Chapter 3).

PROOF. Recall that Equation (3) provides an expression for $V(v_i^t, n)$. Taking the expectation of this expression with respect to v_i^t yields

$$\begin{aligned}\mathbb{E}\left[V(v_i^t,n)\right] &= \int_{-\infty}^{\infty}\left(\left(x - \mathbb{E}\left[Y_2^{(n)}|Y_1^{(n)} = v_i^t\right]\right) G_1^{(n-1)}(x) + \delta W(n-1)\right) f(x)\,dx \\ &= \frac{1}{n}\left(\int_{-\infty}^{\infty} x g_1^{(n)}(x)\,dx - \int_{-\infty}^{\infty}\mathbb{E}\left[Y_2^{(n)}|Y_1^{(n)} = v_i^t\right] g_1^{(n)}(x)\,dx\right) + \delta W(n-1).\end{aligned}$$

Notice, however, that

$$\begin{aligned}&\frac{1}{n}\left(\int_{-\infty}^{\infty} x g_1^{(n)}(x)\,dx - \int_{-\infty}^{\infty}\mathbb{E}\left[Y_2^{(n)}|Y_1^{(n)} = v_i^t\right] g_1^{(n)}(x)\,dx\right) \\ &\qquad = \frac{1}{n}\left(\mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_2^{(n)}\right]\right) = \mathbb{E}\left[Y_1^{(n)}\right] - \mathbb{E}\left[Y_1^{(n-1)}\right].\end{aligned}$$

This is exactly the quantity previously defined as $\widehat{Y}(n)$, implying (as desired) that

$$\mathbb{E}\left[V(v_i^t,n)\right] = \widehat{Y}(n) + \delta W(n-1). \qquad \square$$

With this result in hand, we may rewrite Equation (1) for $n \geq 2$ in terms of W and $\widehat{Y}$ alone:

$$\begin{aligned}W(n+1) = &\frac{1-\delta pq - \delta(1-p)(1-q)}{\delta(1-p)q} W(n) - \frac{p(1-q)}{(1-p)q} W(n-1) \\ &- \frac{p}{\delta(1-p)}\widehat{Y}(n+1) - \frac{p(1-q)}{\delta(1-p)q}\widehat{Y}(n).\end{aligned} \tag{5}$$

Recall, however, that a single buyer remaining from period t will *always* be joined by another buyer at time $t+1$. Thus, when $n = 1$, we have

$$\begin{aligned}W(1) &= p\big(\widehat{Y}(2) + \delta W(1)\big) + (1-p)\delta W(2) \\ &= \frac{p\widehat{Y}(2) + \delta(1-p)W(2)}{1-\delta p}.\end{aligned} \tag{6}$$

Thus, the expected payoff to a buyer is given by a solution to the second-order non-homogenous linear difference equation in Equation (5) and boundary condition in Equation (6). While it is possible to find a solution to this system, the continuous-time limit is significantly more tractable. Recalling that $\delta = e^{-r\Delta}$, $p = \lambda\Delta$, and $q = \rho\Delta$ and taking the limit as Δ goes to zero yields

$$W(n+1) = \frac{r+\lambda+\rho}{\rho}W(n) - \frac{\lambda}{\rho}\left(\widehat{Y}(n) + W(n-1)\right) \text{ for all } n \geq 2, \text{ and} \tag{7}$$

$$W(1) = W(2). \tag{8}$$

We may then rewrite this second-order difference equation as a first-order system of difference equations. In particular, we have, for all $k > 0$,

$$\begin{pmatrix} W(k+2) \\ W(k+1) \end{pmatrix} = \begin{bmatrix} \frac{r+\lambda+\rho}{\rho} & -\frac{\lambda}{\rho} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} W(k+1) \\ W(k) \end{pmatrix} + \begin{pmatrix} -\frac{\lambda}{\rho}\widehat{Y}(k+1) \\ 0 \end{pmatrix}. \tag{9}$$

Note that there exist an infinite number of solutions (in general) to this difference equation; even accounting for the boundary condition in Equation (8), there remains a continuum of possible solutions. However, we are able to rule out solutions in which expected utility diverges to

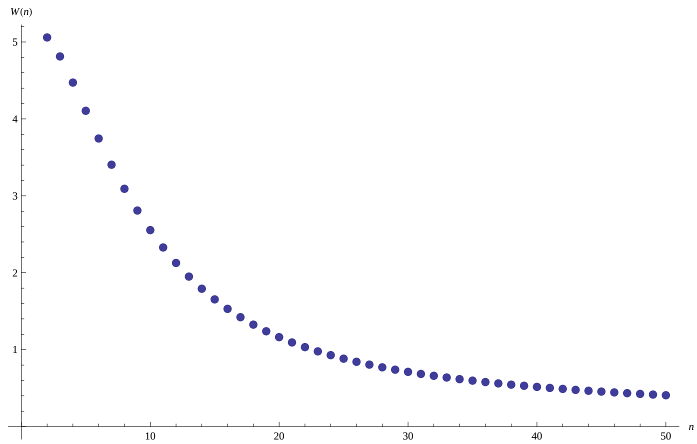


FIGURE 1. $W(n)$ under the exponential distribution, with $r = 0.1$, $\lambda = 1$, and $\rho = 1$.

infinity as the number of buyers on the market grows—there exists a unique bounded (and hence "sensible") solution to the difference equation. To characterize this solution, define

$$\zeta_1 := \frac{r+\lambda+\rho-\sqrt{(r+\lambda+\rho)^2-4\lambda\rho}}{2\rho} \text{ and } \zeta_2 := \frac{r+\lambda+\rho+\sqrt{(r+\lambda+\rho)^2-4\lambda\rho}}{2\rho}.$$

These two constants are the eigenvalues of the "transition" matrix in Equation (9). It is straightforward to show that $\zeta_1\zeta_2 = \frac{\lambda}{\rho}$ and $0 < \zeta_1 < 1 < \zeta_2$ for all $r, \lambda, \rho > 0$.

THEOREM 1 (Equilibrium payoffs with buyer arrivals)**.**
The unique symmetric Markov equilibrium of this infinite-horizon sequential auction game is defined by the ex ante expected payoff function given by

$$W(1) = \frac{\zeta_1\zeta_2}{1-\zeta_1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) \text{ and, for all } n \geq 2, \tag{10}$$

$$W(n) = \zeta_1^{n-1}W(1) + \frac{\zeta_1^n\zeta_2}{\zeta_2-\zeta_1}\sum_{k=1}^{n-1}\left(\zeta_1^{-k}-\zeta_2^{-k}\right)\widehat{Y}(k+1) + \frac{\zeta_1\zeta_2^n-\zeta_1^n\zeta_2}{\zeta_2-\zeta_1}\sum_{k=n}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1), \tag{11}$$

PROOF. The proof may be found in Appendix A. □

We may now calculate the expected payoffs of buyers for each value of $n \in \mathbb{N}$. Figure 1 displays an example of these payoffs when values are drawn from the exponential distribution on the real line. Notice that expected utility is a decreasing function of n: as the number of bidders (and hence competition) increases, each individual bidder's payoff decreases.

Theorem 1 also implies that sellers' revenues may be easily characterized as well. Recall from Lemma 1 that buyers bid their true value less their continuation value. In the continuous time limit, this implies that a buyer with value v_i^t who is one of $n \geq 2$ buyers present will bid

$$b^*(v_i^t, n) = v_i^t - W(n-1).$$

Therefore, a seller facing $n \geq 2$ buyers receives an expected revenue of

$$\Pi(n) := \mathbb{E}\left[Y_2^{(n)}\right] - W(n-1). \tag{12}$$

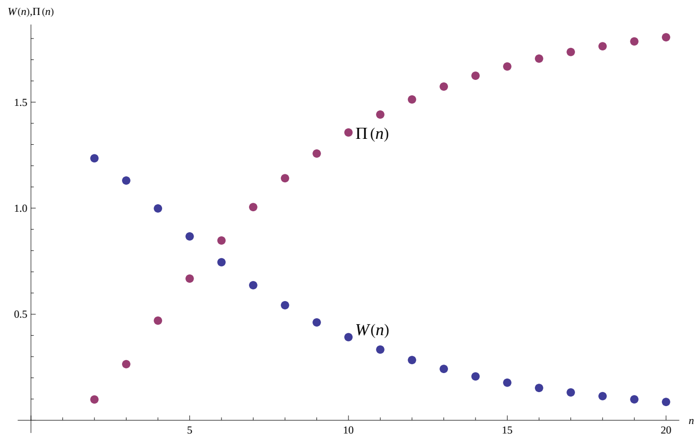


FIGURE 2. $W(n)$ and $\Pi(n)$ with values distributed uniformly on $[1,2]$, with $r = 0.1$, $\lambda = 1$, and $\rho = 1$.

In Figure 2, we plot the expected payoff to buyers *and* sellers as a function of n when buyers' values are distributed according to the uniform distribution on $[1,2]$. Notice that buyers' expected payoffs are decreasing in n, whereas sellers' expected profits are increasing.

These patterns are not simply artifacts of the particular parameterizations. Rather, they hold for all distributions and choices of parameters, as we demonstrate below.

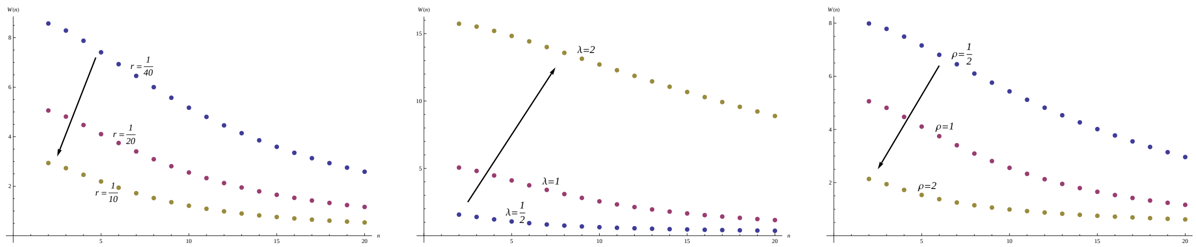
PROPOSITION 1 (Buyer utilities and seller revenues as n changes)**.**
The equilibrium expected payoff to each bidder is declining in the number of buyers present on the market, while the equilibrium expected revenue to each seller is increasing in the number of buyers on the market; for all $n \geq 2$, we have

$$W(n) > W(n+1) \text{ and } \Pi(n) < \Pi(n+1).$$

PROOF. The proof may be found in Appendix A. □

Note that this solution is easily generalized to trading institutions other than the sequential second-price auction. By revenue equivalence, $\widehat{Y}(n)$ is the ex ante expected payoff of a buyer in any standard one-shot auction mechanism with n buyers. Therefore, Equation (11) continues to hold, as is, for markets in which objects are sold via, for instance, sequential first-price auctions—while buyers' bids will differ, their expected payoffs will remain the same, and hence Equation (11) and Equation (12) continue to characterize equilibrium payoffs to both sides of the market.

On the other hand, if a different trading institution were to be employed, then replacing $\widehat{Y}(n)$ by the appropriate ex ante expected payoff of a buyer in that mechanism would yield the corresponding solution for that institution. For example, suppose that each seller employs a (fixed) multi-lateral bargaining game for allocating her object. Letting $\widetilde{Y}(n)$ denote the ex ante expected payoff to each of n buyers from participating in this one-shot bargaining game, equilibrium in the resulting dynamic market is then characterized by the analogue to Equation (11) where $\widehat{Y}$ is replaced by $\widetilde{Y}$.



(A) $r \in \{0.025, 0.05, 0.1\}$ (B) $\lambda \in \{1/2, 1, 2\}$ (C) $\rho \in \{1/2, 1, 2\}$

FIGURE 3. $W(n)$ with exponentially distributed values, and $r = 0.05$, $\lambda = 1$, and $\rho = 1$.

4. COMPARATIVE STATICS

In order to better understand the effects of time and entry on the payoffs of agents in this market, we consider some comparative statics. Figure 3 demonstrates (numerically) the effects of changes in the discount and arrival rates on expected payoffs. One of the advantages, however, of a closed-form solution for $W(n)$ is that clean comparative statics exercises are possible. In particular, we are able to analytically characterize the response of payoffs to changes in each of the main parameters.

PROPOSITION 2 (Comparative Statics)**.**
For all $n \geq 2$,

(1) $\partial W(n)/\partial r < 0 < \partial \Pi(n)/\partial r$;
(2) $\partial W(n)/\partial \lambda > 0 > \partial \Pi(n)/\partial \lambda$; *and*
(3) $\partial W(n)/\partial \rho < 0 < \partial \Pi(n)/\partial \rho$.

PROOF. The proof may be found in Appendix B. □

As expected, a buyer's expected utility increases as they become more patient or as the arrival rate of auctions increases, while it decreases as the arrival rate of additional competition increases. Opposite effects hold when considering seller revenues.

It is also possible to examine the effects of changes in the distribution of values on payoffs. Notice that buyer welfare is an increasing function of $\widehat{Y}(k)$ for all $k \in \mathbb{N}$, and recall that

$$\widehat{Y}(k) = \mathbb{E}\left[Y_1^{(k)}\right] - \mathbb{E}\left[Y_1^{(k-1)}\right] = \frac{1}{k}\left(\mathbb{E}\left[Y_1^{(k)}\right] - \mathbb{E}\left[Y_2^{(k)}\right]\right).$$

Thus, a distributional change that systematically affects this difference will effectively lead to a change in buyer welfare. Notice, however, that replacing F by a distribution $\widetilde{F}$ that stochastically dominates F is not sufficient for increasing $\widehat{Y}$. Although such a change increases both $\mathbb{E}[Y_1^{(k)}]$ and $\mathbb{E}[Y_2^{(k)}]$, it may decrease the difference between the two.[6] Thus, standard stochastic dominance is not sufficient for our purposes, as the ordering of the distributions may be reversed when considering the difference between order statistics.[7]

[6] This difference is referred to by the statistics literature as a "sample spacing." The curious reader is referred to Boland and Shanthikumar (1998) for an overview of stochastic ordering of order statistics, and to Xu and Li (2006) for additional results on the stochastic ordering of sample spacings.

[7] An example of this may reversal may be seen with the distribution functions $F(x) = x$ and $\widetilde{F}(x) = x^2$ on $[0, 1]$. We should point out that the bounded support of these two distributions is *not* essential to this argument.

The statistics literature, however, has identified two different conditions that are sufficient for our purposes. First, Kochar (1999) shows that if $\widetilde{F}$ dominates F in terms of the hazard rate order and either F or $\widetilde{F}$ display a decreasing hazard rate, then $\widehat{Y}(k|\widetilde{F}) \geq \widehat{Y}(k|F)$ for all $k \in \mathbb{N}$. Recall that the hazard rate of a distribution H with density h is given by

$$\lambda_H(t) = \frac{h(t)}{1 - H(t)}.$$

Therefore, $\widetilde{F}$ dominates F in terms of the hazard rate if

$$\lambda_{\widetilde{F}}(t) \leq \lambda_F(t) \text{ for all } t.$$

Note that hazard rate dominance implies first-order stochastic dominance. Second, Bartoszewicz (1986) demonstrates that, if $\widetilde{F}$ dominates F in terms of the dispersive order, then we again have $\widehat{Y}(k|\widetilde{F}) \geq \widehat{Y}(k|F)$ for all $k \in \mathbb{N}$. A distribution $\widetilde{F}$ dominates the distribution F in terms of the dispersive order if, for all $0 \leq x < y \leq 1$,

$$\widetilde{F}^{-1}(y) - \widetilde{F}^{-1}(x) \geq F^{-1}(y) - F^{-1}(x).$$

This is a variability ordering, in the sense that it requires the difference of quantiles of F to be smaller than the difference of the corresponding quantiles of $\widetilde{F}$. Intuitively, when the quantiles of a distribution are more spread out, there is greater variance in the upper tail, and hence a larger difference between the first- and second-order statistics.

5. Markovian Values

Throughout, we have assumed that objects are stochastically equivalent; in effect, this implies that history is irrelevant except for its role in determining the current number of market participants. Now, however, we generalize the model and consider a world in which history *does* matter, although in a manner that allows for a similar form of analysis. In particular, we consider a model in which buyers' values are drawn from one of two different distributions, and the distributions are "persistent" in the sense that the distribution is chosen according to a (known) Markov process. In particular, if the values are drawn from one distribution today, then they are likely to be drawn from the same distribution again tomorrow. In the pure common values example discussed in the introduction, this corresponds to auctioning drilling rights in two different regions, where the choice of region follows a Markov process. Alternately, in a private values setting, this corresponds to "buyer's" and "seller's" markets; one distribution corresponds to the case in which, for some unmodeled exogenous reason, demand (and hence willingness to pay) is higher, independent of the number of current competitors, whereas the second corresponds to the case in which demand is relatively low. As the external forces driving value distributions typically do not change overnight, a buyer's market is likely to persist for some time.

Thus, we consider the case in which there are two states of the world $\{\omega_1, \omega_2\} \in \Omega$. In state ω_i, values are drawn from the distribution F_i (with corresponding density f_i). The state of the world is assumed to be commonly known in each period. In addition, the (symmetric) probability of transitioning from one state to the other is $\tau = \pi\Delta, \pi > 0$. Note that the model is easily generalized to a greater number of states or asymmetric transitions; doing so would, however, greatly complicate notation and explication while providing little in the way of additional insight.

We will denote by $W(n, \omega_i)$ the expected payoff to a buyer when the state of the world is $\omega_i \in \Omega$ and there are n total buyers present in the market; recall that this is an ex ante payoff, as this is before a seller has arrived on the market in the period and the buyers do not yet know their valuations. We will denote by $V(v_i^t, n, \omega)$ the expected payoff to a buyer when a seller has arrived, and hence the buyer knows her value v_i^t, and, slightly abusing notation, we will write $\mathbb{E}[V(v_i^t, n, \omega_i)]$ to denote the expected value to a buyer when a seller is present but the buyer does not (yet) know her value; the expectation is taken with respect to the distribution F_i.

Thus, for $i \in \{1, 2\}$, we have

$$W(n, \omega_i) = pq\mathbb{E}[V(v_i^t, n+1, \omega_i)] + p(1-q)\mathbb{E}[V(v_i^t, n, \omega_i)] + \delta(1-p)\begin{bmatrix}(1-\tau)qW(n+1, \omega_i) + \tau q W(n+1, \omega_{-i}) \\ +(1-\tau)(1-q)W(n, \omega_i) + \tau(1-q)W(n, \omega_{-i})\end{bmatrix}.$$

Furthermore, it is relatively straightforward (using the same methods as in previous sections) to see that Lemma 1 again applies in this setting—a buyer's optimal behavior is to bid her true value less her continuation value. This implies that

$$\mathbb{E}[V(v_i^t, k, \omega_i)] = \widehat{Y}(k, \omega_i) + \delta\left[(1-\tau)W(k-1, \omega_i) + \tau W(k-1, \omega_{-i})\right].$$

Combining these two expressions leads to a system of (coupled) second-order difference equations. Once again, the arithmetic becomes cumbersome due to the interaction of the various parameters; therefore, we pass to the continuous time limit as Δ approaches zero. This yields

$$\begin{aligned} W(n+2, \omega_i) = &\frac{r+\pi+\lambda+\rho}{\rho}W(n+1, \omega_i) - \frac{\pi}{\rho}W(n+1, \omega_{-i}) \\ &- \frac{\lambda}{\rho}\widehat{Y}(n+1, \omega_i) - \frac{\lambda}{\rho}W(n, \omega_i) \text{ for all } n > 1. \end{aligned} \tag{13}$$

Notice that this expression is, as in previous sections, a difference equation that is linear and second-order in $W(\cdot, \omega_i)$. However, $W(\cdot, \omega_{-i})$ also appears in the equation, implying that we have a *coupled* system of difference equations.

As in the previous section, we are interested in finding a solution to this system of difference equations. Moreover, we require that this solution be bounded, as well as that the solution satisfies the appropriate boundary conditions. These boundary conditions may be found in a manner analogous to those discussed previously. Specifically, a buyer who is alone on the market will be immediately joined by another buyer, regardless of the current underlying market state. This leads to, in the limit as Δ approaches zero,

$$W(1, \omega_i) = W(2, \omega_i) \text{ for } i \in \{1, 2\}. \tag{14}$$

Once again, this system of difference equations has a continuum of solutions when we ignore the boundedness constraint; however, we can show, in a manner similar to that of the previous sections, that there exists a *unique* bounded solution that satisfies the boundary conditions of

Equation (14). To characterize this solution, define

$$\xi_1 = \frac{r+\lambda+\rho+2\pi - \sqrt{(r+\lambda+\rho+2\pi)^2 - 4\lambda\rho}}{2\rho} \text{ and}$$
$$\xi_2 = \frac{r+\lambda+\rho+2\pi + \sqrt{(r+\lambda+\rho+2\pi)^2 - 4\lambda\rho}}{2\rho},$$

as well as

$$\xi_3 = \frac{r+\lambda+\rho - \sqrt{(r+\lambda+\rho)^2 - 4\lambda\rho}}{2\rho} = \zeta_1 \text{ and}$$
$$\xi_4 = \frac{r+\lambda+\rho + \sqrt{(r+\lambda+\rho)^2 - 4\lambda\rho}}{2\rho} = \zeta_2.$$

These are the eigenvalues of the transition matrix behind the coupled system of difference equations. Note that $0 < \xi_1, \xi_3 < 1 < \xi_2, \xi_4$ and $\xi_1\xi_2 = \frac{\lambda}{\rho} = \xi_3\xi_4$. We are then able to characterize the symmetric Markov equilibrium of this sequence of auctions.

THEOREM 2 (Equilibrium payoffs with Markovian values).
The unique symmetric equilibrium with bounded payoffs of this infinite-horizon sequential auction game is determined by the ex ante expected payoff functions given by, for $i = 1, 2$,

$$\begin{aligned} W(n,\omega_i) = {} & \xi_1^{n-1}\left(\frac{W(1,\omega_i) - W(1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_1^n\xi_2}{\xi_2-\xi_1}\left(\sum_{k=1}^{n-1}\left(\xi_1^{-k} - \xi_2^{-k}\right)\frac{\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_1\xi_2^n - \xi_1^n\xi_2}{\xi_2-\xi_1}\left(\sum_{k=n}^{\infty}\xi_2^{-k}\frac{\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \xi_3^{n-1}\left(\frac{W(1,\omega_i) + W(1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_3^n\xi_4}{\xi_4-\xi_3}\left(\sum_{k=1}^{n-1}\left(\xi_3^{-k} - \xi_4^{-k}\right)\frac{\widehat{Y}(k+1,\omega_i) + \widehat{Y}(k+1,\omega_{-i})}{2}\right) \\ & + \frac{\xi_3\xi_4^n - \xi_3^n\xi_4}{\xi_4-\xi_3}\left(\sum_{k=n}^{\infty}\xi_4^{-k}\frac{\widehat{Y}(k+1,\omega_i) + \widehat{Y}(k+1,\omega_{-i})}{2}\right), \end{aligned} \tag{15}$$

for all $n \geq 2$, where $W(1,\omega_i)$ for $i = 1, 2$ is given by

$$\begin{aligned} W(1,\omega_i) = {} & \frac{\xi_1\xi_2}{1-\xi_1}\sum_{k=1}^{\infty}\xi_4^{-k}\frac{\widehat{Y}(k+1,\omega_i) + \widehat{Y}(k+1,\omega_{-i})}{2} \\ & + \frac{\xi_3\xi_4}{1-\xi_3}\sum_{k=1}^{\infty}\xi_2^{-k}\frac{\widehat{Y}(k+1,\omega_i) - \widehat{Y}(k+1,\omega_{-i})}{2}. \end{aligned} \tag{16}$$

PROOF. The proof may be found in Appendix A. □

It is straightforward to show (due to the closed-form expression above) that $W(n,\omega_i)$ is decreasing in n. Moreover, the comparative statics results of Section 4 continue to hold—buyers' expected

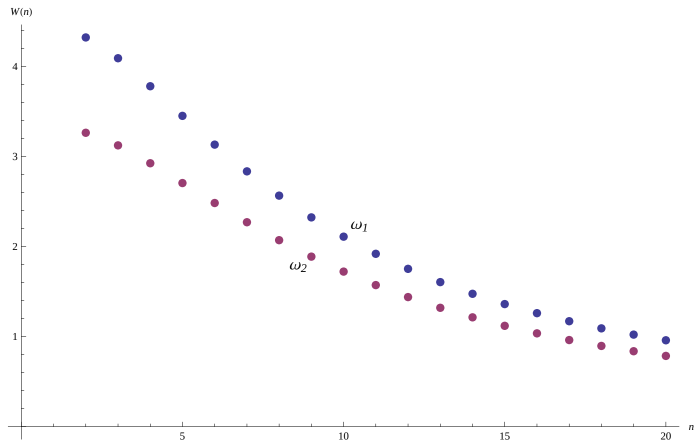


FIGURE 4. $W(n, \omega_i)$ with $F_1(v) = 1 - e^{-v}$, $F_2(v) = 1 - e^{-2v}$, $r = 0.05$, $\lambda = 1$, $\rho = 1$, and $\pi = 0.05$.

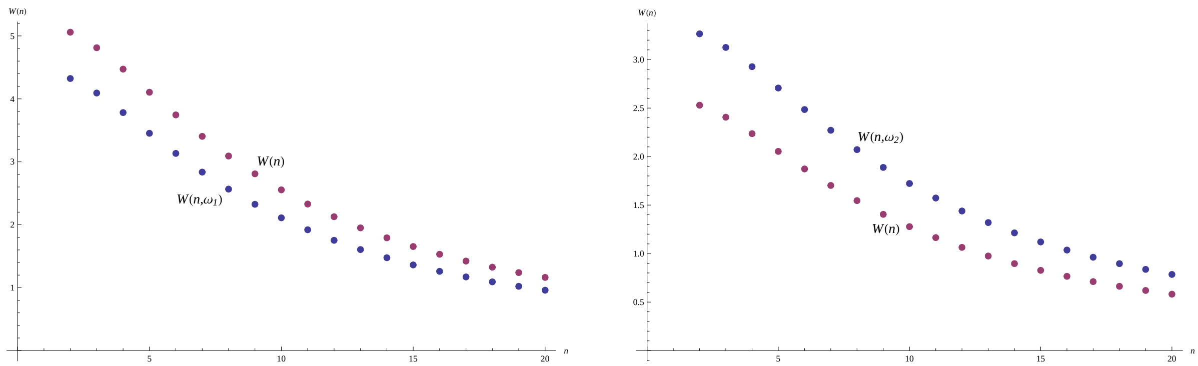
payoffs are decreasing in the discount rate and the arrival rate of new buyers, but increasing in the arrival rate of new objects.

Also, note that when $\pi = 0$ or the distributions F_1 and F_2 are identical, the solution above collapses to Equation (11); we return to the case studied in Section 3. Moreover, notice that the difference in expected payoffs between the two states is determined solely by the difference in sample spacings across the two states. Note that

$$W(n, \omega_i) - W(n, \omega_{-i}) = \xi_1^{n-1} \left(W(1, \omega_1) - W(1, \omega_{-i}) \right) + \frac{\xi_1^n \xi_2}{\xi_2 - \xi_1} \left(\sum_{k=1}^{n-1} \left(\xi_1^{-k} - \xi_2^{-k} \right) \left(\widehat{Y}(k+1, \omega_i) - \widehat{Y}(k+1, \omega_{-i}) \right) \right) + \frac{\xi_1 \xi_2^n - \xi_1^n \xi_2}{\xi_2 - \xi_1} \left(\sum_{k=n}^{\infty} \xi_2^{-k} \left(\widehat{Y}(k+1, \omega_i) - \widehat{Y}(k+1, \omega_{-i}) \right) \right).$$

Whenever this difference is positive, buyers strictly prefer to be in state ω_i than in state ω_{-i}; the converse of this is, of course, that bidding is more aggressive in state ω_i than in state ω_{-i} in the sense of absolute magnitude of bid shading away from the true values.[8] Figure 4 demonstrates an example of this in the case that F_1 dominates F_2 in terms of the dispersive order as discussed earlier. Note that $W(n, \omega_1) > W(n, \omega_2)$ for all n. Moreover, as may be seen in Figure 5, buyer's payoffs in state ω_1 are lower than they would be if there were no transitions to state ω_2, while the payoffs in state ω_2 are higher than they would otherwise be in a one-state model. In essence, the possibility of transitioning to an unambiguously better state improves buyer utility in state ω_2, while the possibility of transitioning to an unambiguously worse state decreases buyer utility in state ω_1.

[8] Note that since $\partial \xi_1 / \partial \pi < 0 < \partial \xi_2 / \partial \pi$, as the rate of state-to-state transitions increases without bound, there is enough churning between the two states that the differential between the state-contingent payoffs goes to zero.



(A) State ω_1 (B) State ω_2

FIGURE 5. $W(n,\omega_i)$ and $W(n)$ with distribution F_i, $r = 0.05$, $\lambda = 1$, $\rho = 1$, and $\pi = 0.05$.

6. CONCLUSION

This paper characterizes the manner in which current market conditions, as well as anticipated future conditions, create an endogenous option value for bidders in a dynamic market. Since buyers must trade off purchasing in the present against participating in the future, the value of this future option is crucial for current-period bidding; however, the value of the option is itself determined by equilibrium bidding behavior. We show that this endogenous option value is, in fact, the expected discounted sum of the potential payoffs from individual transactions in the infinite sequence of possible states of the world, each differentiated by the potential number of buyers present on the market at that time. When the trading institution is an auction mechanism, buyers are therefore willing to bid their true values less the discounted option value of participating in this future sequence of auctions.

There are several directions for extending our analysis. One possibility is dropping the assumption of stochastic equivalence and endowing each buyer with a fixed private value for obtaining each object. Zeithammer (2009) explores this question when the values may differ across objects. There are several technical difficulties in conducting such an analysis in a model with sealed-bid auctions when values are perfectly correlated across objects, however. These complications are discussed in Said (2010). In particular, the sequential second-price auction is not efficient in the presence of buyer arrivals, so instead we examine a model in which objects are sold using the ascending auction format. Another potentially interesting line of research involves allowing for multiple simultaneous auctions, or allowing sellers to remain on the market for several periods and overlapping with one another. Additional possibilities include endogenizing the entry behavior of buyers and sellers in response to market conditions and dynamics, or allowing for the setting of reserve prices by sellers; in particular, considering the limit behavior of a model with a cap on the number of market participants may be particularly useful in understanding behavior with reserve prices. These extensions are, however, left for future work.

Appendix A. Omitted Proofs

Proof of Theorem 1. Define

$$w_m := (W(m+1), W(m))' \text{ and } y_m := \left(-\zeta_1\zeta_2\widehat{Y}(m+1), 0\right)'$$

for all $m \in \mathbb{N}$. Then Equation (9) becomes

$$w_{n+1} = Aw_n + y_n,$$

where A is the matrix in Equation (9). Applying Elayedi (2005, Theorem 3.17) yields the general solution

$$w_n = A^{n-1}w_1 + \sum_{k=1}^{n-1} A^{n-k-1}y_k.$$

Recalling that ζ_1 and ζ_2 are the eigenvalues of the matrix A, it is straightforward to show that

$$A^k = \frac{1}{\zeta_2 - \zeta_1}\begin{bmatrix} \zeta_2^{k+1} - \zeta_1^{k+1} & \zeta_1^{k+1}\zeta_2 - \zeta_1\zeta_2^{k+1} \\ \zeta_2^k - \zeta_1^k & \zeta_1^k\zeta_2 - \zeta_1\zeta_2^k \end{bmatrix},$$

implying that the general solution to this second-order system may be (after some rearrangement) written as

$$\begin{aligned} W(n) = {} & \frac{\zeta_2^{n-1}}{\zeta_2 - \zeta_1}\left(W(2) - \zeta_1 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_2^{-k}\widehat{Y}(k+1)\right) \\ & - \frac{\zeta_1^{n-1}}{\zeta_2 - \zeta_1}\left(W(2) - \zeta_2 W(1) - \zeta_1\zeta_2 \sum_{k=1}^{n-1} \zeta_1^{-k}\widehat{Y}(k+1)\right). \end{aligned} \tag{A.1}$$

Consider the second term in the above expression. Since $0 < \zeta_1 < 1$, the first two parts of it are clearly bounded; in particular, we have $\zeta_1^{n-1}(\zeta_2 - \zeta_1)^{-1}(W(2) - \zeta_2 W(1)) \to 0$ as $n \to \infty$. The third part of this term may be rewritten as

$$\begin{aligned} \frac{\zeta_1^n\zeta_2}{\zeta_2 - \zeta_1}\sum_{k=1}^{n-1}\zeta_1^{-k}\widehat{Y}(k+1) &= \frac{\zeta_2}{\zeta_2 - \zeta_1}\sum_{k=1}^{n-1}\zeta_1^k\widehat{Y}(n-k+1) \\ &\leq \frac{\zeta_2}{\zeta_2 - \zeta_1}\sum_{k=1}^{n-1}\zeta_1^k\sigma < \frac{\zeta_2}{\zeta_2 - \zeta_1}\sum_{k=1}^{\infty}\zeta_1^k\sigma < \frac{\zeta_1\zeta_2\sigma}{(\zeta_2 - \zeta_1)(1 - \zeta_1)}, \end{aligned}$$

where σ^2 is the (assumed finite) variance of the distribution F. This follows from Arnold and Groeneveld (1979), who show that

$$\mathbb{E}\left[Y_1^{(m)}\right] - \mathbb{E}\left[Y_2^{(m)}\right] \leq \frac{m\sigma}{\sqrt{m-1}} \text{ for all } m > 1.$$

Recalling the definition of $\widehat{Y}$, we then have

$$\widehat{Y}(m) \leq \frac{\sigma}{\sqrt{m-1}} < \sigma$$

for all $m > 1$, implying that the second term in Equation (A.1) is bounded.

The first term in Equation (A.1), however, is multiplied by positive powers of $\zeta_2 > 1$, implying that an appropriate choice of $W(2)$ is crucial for ensuring the boundedness of our solution. One such choice is to let

$$W(2) = \zeta_1 W(1) + \zeta_1\zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k}\widehat{Y}(k+1). \tag{A.2}$$

Note that, for any $W(1) \in \mathbb{R}$, $W(2)$ is well-defined by the expression above, as $\zeta_2 > 1$ and $\{\widehat{Y}(m)\}$ is a bounded sequence. The first term in Equation (A.1) then becomes

$$\begin{aligned}\frac{\zeta_1\zeta_2^n}{\zeta_2-\zeta_1}\sum_{k=n}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) &= \frac{\zeta_1}{\zeta_2-\zeta_1}\sum_{k=n}^{\infty}\zeta_2^{n-k}\widehat{Y}(k+1)\\ &= \frac{\zeta_1}{\zeta_2-\zeta_1}\sum_{k=0}^{\infty}\zeta_2^{-k}\widehat{Y}(n+k+1) < \infty,\end{aligned}$$

where we again rely on the boundedness of the sequence $\{\widehat{Y}(m)\}$ when F has finite variance. Thus, for any choice of $W(1)$, choosing $W(2)$ in accordance with Equation (A.2) leads to a bounded solution of the difference equation.

To show that this is the *unique* bounded solution, consider any fixed $W(1)$, and denote by $\bar{c}$ the choice of $W(2)$ corresponding to Equation (A.2). Fix any arbitrary $\alpha \in \mathbb{R}$, and let $W(2) = \alpha\bar{c}$. Then, denoting by $\bar{W}$ the solution when $W(2) = \bar{c}$, Equation (A.1) becomes

$$\begin{aligned}W(n) &= \frac{\zeta_2^{n-1}}{\zeta_2-\zeta_1}\left(\alpha\bar{c} - \zeta_1 W(1) - \zeta_1\zeta_2\sum_{k=1}^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right)\\ &\quad - \frac{\zeta_1^{n-1}}{\zeta_2-\zeta_1}\left(\alpha\bar{c} - \zeta_2 W(1) - \zeta_1\zeta_2\sum_{k=1}^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)\right)\\ &= \frac{\zeta_2^{n-1}}{\zeta_2-\zeta_1}\left(\bar{c} + (\alpha-1)\bar{c} - \zeta_1 W(1) - \zeta_1\zeta_2\sum_{k=1}^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right)\\ &\quad - \frac{\zeta_1^{n-1}}{\zeta_2-\zeta_1}\left(\bar{c} + (\alpha-1)\bar{c} - \zeta_2 W(1) - \zeta_1\zeta_2\sum_{k=1}^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)\right)\\ &= \bar{W}(n) + \left(\zeta_2^{n-1} - \zeta_1^{n-1}\right)\frac{(\alpha-1)}{\zeta_2-\zeta_1}\bar{c}.\end{aligned}$$

Since $\zeta_2 > 1 > \zeta_1 > 0$, the above expression remains bounded if, and only if, $\alpha = 1$. Note that we also have boundedness for arbitrary α if $\bar{c} = 0$. However, this would imply that $W(1) = -\zeta_2\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) < 0$, contradicting the boundary condition in Equation (8).

Thus, for any choice of $W(1)$, choosing $W(2)$ in accordance with Equation (A.2) leads to a bounded solution. The only remaining free variable is then $W(1)$, which is then determined by the boundary condition derived from single-buyer behavior; that is, $W(1)$ may be found by combining Equation (8) and Equation (A.2), leading to the condition stated in the proposition, as desired. □

PROOF OF PROPOSITION 1. Note that, by using Equation (11) from Theorem 1, the difference between $W(n)$ and $W(n+1)$, for arbitrary $n \geq 2$, may be written as

$$
\begin{aligned}
&W(n) - W(n+1) = \\
&= \zeta_1^{n-1} W(1) + \frac{\zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \left(\zeta_1^{-k} - \zeta_2^{-k}\right) \widehat{Y}(k+1) + \frac{\zeta_1 \zeta_2^n - \zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) \\
&\quad - \zeta_1^n W(1) - \frac{\zeta_1^{n+1} \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n} \left(\zeta_1^{-k} - \zeta_2^{-k}\right) \widehat{Y}(k+1) - \frac{\zeta_1 \zeta_2^{n+1} - \zeta_1^{n+1} \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=n+1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) \\
&= (1-\zeta_1)\zeta_1^{n-1} W(1) + (1-\zeta_1) \frac{\zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=1}^{n-1} \left(\zeta_1^{-k} - \zeta_2^{-k}\right) \widehat{Y}(k+1) \\
&\quad + (1-\zeta_1) \frac{\zeta_1 \zeta_2^n - \zeta_1^n \zeta_2}{\zeta_2 - \zeta_1} \sum_{k=n}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) - \zeta_1 \zeta_2^n \sum_{k=n}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) \\
&= (1-\zeta_1) W(n) - \zeta_1 \zeta_2^n \sum_{k=n}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1). \qquad \text{(A.3)}
\end{aligned}
$$

This implies that, for arbitrary $m \geq 2$,

$$
W(m) = \zeta_1 W(m-1) + \zeta_1 \sum_{k=0}^{\infty} \zeta_2^{-k} \widehat{Y}(m+k) \text{ and} \qquad \text{(A.4)}
$$

$$
W(m+1) = \zeta_1 W(m) + \zeta_1 \sum_{k=0}^{\infty} \zeta_2^{-k} \widehat{Y}(m+k+1).
$$

Taking the difference between these two expressions yields

$$
W(m) - W(m+1) = \zeta_1 \left(W(m-1) - W(m)\right) + \zeta_1 \sum_{k=0}^{\infty} \zeta_2^{-k} \left(\widehat{Y}(m+k) - \widehat{Y}(m+k+1)\right).
$$

Recall that $\widehat{Y}(\cdot)$ is decreasing in its argument, $\zeta_1 > 0$, and $\zeta_2 > 0$. Thus, $W(m-1) \geq W(m)$ is sufficient to show that $W(m) > W(m+1)$. Since $W(1) = W(2)$, this is true for $m = 2$. Proceeding inductively, we have $W(n) > W(n+1)$ for all $n \geq 2$.

Finally, recall from Equation (12) that $\Pi(n) := \mathbb{E}[Y_2^{(n)}] - W(n-1)$. Since $\mathbb{E}\left[Y_2^{(n)}\right]$ is increasing in n, the fact that $\Pi(n)$ is also increasing in n follows immediately from the above. □

PROOF OF THEOREM 2. Letting $a := \frac{r+\pi+\lambda+\rho}{\rho}$, $b := -\frac{\pi}{\rho}$, and $c := -\frac{\lambda}{\rho}$, we may then write the coupled system defined by Equation (13) as

$$
\begin{pmatrix} W_1(n+2) \\ W_2(n+2) \\ W_1(n+1) \\ W_2(n+1) \end{pmatrix} = \begin{bmatrix} a & b & c & 0 \\ b & a & 0 & c \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} W_1(n+1) \\ W_2(n+1) \\ W_1(n) \\ W_2(n) \end{pmatrix} + \begin{pmatrix} c\widehat{Y}_1(n+1) \\ c\widehat{Y}_2(n+1) \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(A.5)}
$$

where the subscripts on W and $\widehat{Y}$ denote the state of the world. Writing A for the matrix of coefficients above, and letting

$$
w_k := (W_1(k+1), W_2(k+1), W_1(k), W_2(k))'
$$

and

$$y_k := (c\widehat{Y}_1(k+1), c\widehat{Y}_2(k+1), 0, 0)',$$

we can write Equation (A.5) more compactly as

$$w_{n+1} = Aw_n + y_n.$$

Applying Elayedi (2005, Theorem 3.17), we then may conclude that the general solution to this system is

$$w_n = A^{n-1}w_1 + \sum_{k=1}^{n-1} A^{n-k-1}y_k. \tag{A.6}$$

In particular, if we denote by $a_{ij}^{(k)}$ the ij-th element of A^k, this may be rewritten as

$$\begin{aligned}\begin{pmatrix} W_1(n) \\ W_2(n) \end{pmatrix} &= \begin{bmatrix} a_{31}^{(n-1)} & a_{32}^{(n-1)} & a_{33}^{(n-1)} & a_{34}^{(n-1)} \\ a_{41}^{(n-1)} & a_{42}^{(n-1)} & a_{43}^{(n-1)} & a_{34}^{(n-1)} \end{bmatrix} \begin{pmatrix} W_1(2) \\ W_2(2) \\ W_1(1) \\ W_2(1) \end{pmatrix} \\ &\quad + c\sum_{k=1}^{n-1} \begin{bmatrix} a_{31}^{(n-k-1)} & a_{32}^{(n-k-1)} \\ a_{41}^{(n-k-1)} & a_{42}^{(n-k-1)} \end{bmatrix} \begin{pmatrix} \widehat{Y}_1(k+1) \\ \widehat{Y}_2(k+1) \end{pmatrix}.\end{aligned} \tag{A.7}$$

Note that A is diagonalizable: defining $D := \text{diag}\,[\xi_1, \xi_2, \xi_3, \xi_4]$ and $P := [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$ to be the diagonal matrix of eigenvalues of A the matrix formed from the corresponding eigenvectors, respectively, we may write $A = PDP^{-1}$. Therefore, $A^k = PD^kP^{-1}$, allowing for the explicit calculation of A^k for all k. In particular, we have

$$P = \begin{bmatrix} -\xi_1 & -\xi_2 & \xi_3 & \xi_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which implies that

$$\begin{aligned} a_{31}^{(k)} = a_{42}^{(k)} &= \frac{\xi_2^k - \xi_1^k}{2(\xi_2 - \xi_1)} + \frac{\xi_4^k - \xi_3^k}{2(\xi_4 - \xi_3)}, \\ a_{33}^{(k)} = a_{44}^{(k)} &= \frac{\xi_1\xi_2^k - \xi_1^k\xi_2}{2(\xi_2 - \xi_1)} + \frac{\xi_3\xi_4^k - \xi_3^k\xi_4}{2(\xi_4 - \xi_3)}, \\ a_{32}^{(k)} = a_{41}^{(k)} &= \frac{\xi_1^k - \xi_2^k}{2(\xi_2 - \xi_1)} + \frac{\xi_4^k - \xi_3^k}{2(\xi_4 - \xi_3)}, \\ a_{34}^{(k)} = a_{43}^{(k)} &= \frac{\xi_1^k\xi_2 - \xi_1\xi_2^k}{2(\xi_2 - \xi_1)} + \frac{\xi_3\xi_4^k - \xi_3^k\xi_4}{2(\xi_4 - \xi_3)}. \end{aligned}$$

Notice that (due to the symmetry detailed above), we need concentrate only on the value function from one state. Thus, we may (after some rearrangement) write $W_1(n)$ as

$$\begin{aligned}&\frac{\xi_2^{n-1}}{\xi_2-\xi_1}\left(\frac{W_1(2)-W_2(2)}{2}-\xi_1\frac{W_1(1)-W_2(1)}{2}-\xi_1\xi_2\sum_{k=1}^{n-1}\xi_2^{-k}\frac{\widehat{Y}_1(k+1)-\widehat{Y}_2(k+1)}{2}\right)\\&\quad-\frac{\xi_1^{n-1}}{\xi_2-\xi_1}\left(\frac{W_1(2)-W_2(2)}{2}-\xi_2\frac{W_1(1)-W_2(1)}{2}-\xi_1\xi_2\sum_{k=1}^{n-1}\xi_1^{-k}\frac{\widehat{Y}_1(k+1)-\widehat{Y}_2(k+1)}{2}\right)\\&\quad+\frac{\xi_4^{n-1}}{\xi_4-\xi_3}\left(\frac{W_1(2)+W_2(2)}{2}-\xi_3\frac{W_1(1)+W_2(1)}{2}-\xi_3\xi_4\sum_{k=1}^{n-1}\xi_4^{-k}\frac{\widehat{Y}_1(k+1)+\widehat{Y}_2(k+1)}{2}\right)\\&\quad-\frac{\xi_3^{n-1}}{\xi_4-\xi_3}\left(\frac{W_1(2)+W_2(2)}{2}-\xi_4\frac{W_1(1)+W_2(1)}{2}-\xi_3\xi_4\sum_{k=1}^{n-1}\xi_3^{-k}\frac{\widehat{Y}_1(k+1)+\widehat{Y}_2(k+1)}{2}\right).\end{aligned}$$

Since $0<\xi_1,\xi_3<1$ and both F and G are assumed to have finite variance, it is straightforward to verify that for any choices of $W_1(1), W_1(2), W_2(1)$, and $W_2(2)$ that the second and fourth terms in this expression are bounded. As in the proof of Theorem 1, however, the fact that $\xi_2,\xi_4>1$ implies that the first and third terms may be unbounded if $W_1(2)$ and $W_2(2)$ are not chosen carefully. Therefore, let

$$\begin{aligned}W_1(2) :=\ & \frac{\xi_1+\xi_3}{2}W_1(1)+\frac{\xi_3-\xi_1}{2}W_2(1)\\&+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_2^{-k}+\xi_4^{-k}}{2}\widehat{Y}_1(k+1)+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}-\xi_2^{-k}}{2}\widehat{Y}_2(k+1)\ \text{ and}\end{aligned}\tag{A.8}$$

$$\begin{aligned}W_2(2) :=\ & \frac{\xi_3+\xi_1}{2}W_2(1)+\frac{\xi_3-\xi_1}{2}W_1(1)\\&+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}+\xi_2^{-k}}{2}\widehat{Y}_2(k+1)+\frac{\lambda}{\rho}\sum_{k=1}^{\infty}\frac{\xi_4^{-k}-\xi_2^{-k}}{2}\widehat{Y}_1(k+1).\end{aligned}\tag{A.9}$$

Verifying that these values lead to a bounded solution for $W_1(n)$ and $W_2(n)$ for any choices of $W_1(1)$ and $W_2(1)$ follows in a manner directly analogous to that used in the proof of Theorem 1.

Finally, the values of $W_1(1)$ and $W_2(1)$ are given by the joint solution to the system of equations derived by combining the expressions for $W_1(2)$ and $W_2(2)$ above with the boundary condition from Equation (14). This immediately yields the expression found in Equation (16).

Uniqueness of the solution to the system of difference equations is shown in exactly the same manner as in the proof of Theorem 1. Thus, after some arithmetic manipulation and rearrangement, the unique bounded solution is given, as desired, by Equation (15). □

APPENDIX B. PROOF OF PROPOSITION 2

We prove each claim of Proposition 2 separately.[9] We begin by showing that $\partial W(n)/\partial r < 0$.

CLAIM. *For all $n \geq 2$, $\partial W(n)/\partial r < 0 < \partial \Pi(n)/\partial r$.*

PROOF OF CLAIM. Note that we may write $W(n)$ as

$$W(n) = \frac{\zeta_1\zeta_2(\zeta_2-1)}{(\zeta_2-\zeta_1)(1-\zeta_1)}\left[\zeta_1^{n-1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1)\right] + \frac{\zeta_1\zeta_2}{(\zeta_2-\zeta_1)}\left[\zeta_1^{n-1}\sum_{k=1}^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)+\zeta_2^{n-1}\sum_{k=n}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1)\right].$$

In addition, notice that

$$\frac{\partial\zeta_1}{\partial r} = -\frac{\zeta_1}{\rho(\zeta_2-\zeta_1)} < 0 \text{ and } \frac{\partial\zeta_2}{\partial r} = \frac{\zeta_2}{\rho(\zeta_2-\zeta_1)} > 0.$$

Therefore, letting $C_1 := \frac{\zeta_1\zeta_2}{\rho(\zeta_2-\zeta_1)^2} > 0$, we may write

$$\begin{aligned}
\frac{\partial W(n)}{\partial r} &= C_1\frac{(\zeta_2-1)}{(1-\zeta_1)}\left[-\sum_{k=1}^{\infty}(n-1)\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)-\sum_{k=1}^{\infty}k\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right]\\
&\quad + C_1\left[\frac{\zeta_2}{(1-\zeta_1)}-\frac{(\zeta_2-1)(\zeta_1+\zeta_2)}{(\zeta_2-\zeta_1)(1-\zeta_1)}-\frac{\zeta_1(\zeta_2-1)}{(1-\zeta_1)^2}\right]\left[\sum_{k=1}^{\infty}\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right]\\
&\quad + C_1\left[-\sum_{k=1}^{n-1}(n-1-k)\zeta_1^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)+\sum_{k=n}^{\infty}(n-1-k)\zeta_2^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right]\\
&\quad - C_1\frac{(\zeta_1+\zeta_2)}{(\zeta_2-\zeta_1)}\left[\sum_{k=1}^{n-1}\zeta_1^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)+\sum_{k=n}^{\infty}\zeta_2^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right].
\end{aligned}$$

Since $n \geq 2$, $C_1 > 0$, and $\zeta_2 > 1 > \zeta_1 > 0$, it is easy to see that

$$C_1\frac{(\zeta_2-1)}{(1-\zeta_1)}\left[-\sum_{k=1}^{\infty}(n-1)\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)-\sum_{k=1}^{\infty}k\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right] < 0.$$

Thus, the first line in the expression above for $\partial W(n)/\partial r$ is negative. Similarly,

$$-C_1\left[\sum_{k=1}^{n-1}(n-1-k)\zeta_1^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)-\sum_{k=n}^{\infty}(n-1-k)\zeta_2^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\right] < 0$$

since the first summand is multiplied by $(n-1-k) \geq 0$ for all $k < n$ and the second summand is multiplied by $(n-1-k) < 0$ for all $k \geq n$. Note further that

$$\begin{aligned}
\sum_{k=1}^{n-1}\zeta_1^{n-1}\zeta_1^{-k}\widehat{Y}(k+1)+\sum_{k=n}^{\infty}\zeta_2^{n-1}\zeta_2^{-k}\widehat{Y}(k+1) &> \sum_{k=1}^{n-1}\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)+\sum_{k=n}^{\infty}\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1)\\
&= \sum_{k=1}^{\infty}\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1),
\end{aligned}$$

[9] Note that, as is obvious from Equation (12), that $\partial\Pi(n)/\partial x = -\partial W(n-1)/\partial x$ for all $n \geq 2$, where $x \in \{r, \lambda, \rho\}$.

and that

$$\frac{\zeta_2}{(1-\zeta_1)} - \frac{(\zeta_2-1)(\zeta_1+\zeta_2)}{(\zeta_2-\zeta_1)(1-\zeta_1)} - \frac{\zeta_1(\zeta_2-1)}{(1-\zeta_1)^2} - \frac{(\zeta_1+\zeta_2)}{(\zeta_2-\zeta_1)} = -\frac{\zeta_1}{(1-\zeta_1)^2}.$$

Thus, the sum of the second and fourth lines of the expression above for $\partial W(n)/\partial r$ is smaller than

$$-C_1\frac{\zeta_1}{(1-\zeta_1)^2}\sum_{k=1}^{\infty}\zeta_1^{n-1}\zeta_2^{-k}\widehat{Y}(k+1) < 0.$$

Therefore, the sum of the four lines in the expression above for $\partial W(n)/\partial r$ is a sum of four negative terms, implying that $\partial W(n)/\partial r < 0$, as desired. □

In order to show that $\partial W(n)/\partial\lambda > 0$ for arbitrary n, we first need to show that this holds for $n = 1$.

CLAIM. $\partial W(1)/\partial\lambda > 0$.

PROOF OF CLAIM. Recall from Equation (10) that

$$W(1) = \frac{\zeta_1\zeta_2}{1-\zeta_1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1).$$

Let ζ_1' denote $\partial\zeta_1/\partial\lambda$ and ζ_2' denote $\partial\zeta_2/\partial\lambda$, and note that $\zeta_1' > 0$ and $\zeta_2' > 0$. We may write

$$\begin{aligned}\frac{\partial W(1)}{\partial\lambda} &= \frac{\zeta_1\zeta_2}{1-\zeta_1}\sum_{k=1}^{\infty}-k\zeta_2^{-k-1}\zeta_2'\widehat{Y}(k+1) + \left(\frac{\zeta_1\zeta_2'+\zeta_1'\zeta_2}{1-\zeta_1} + \frac{\zeta_1\zeta_2\zeta_1'}{(1-\zeta_1)^2}\right)\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1)\\ &= \frac{\zeta_1\zeta_2'+\zeta_1'\zeta_2}{1-\zeta_1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) + \frac{\zeta_1\zeta_2'}{1-\zeta_1}\sum_{k=1}^{\infty}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k+1), \qquad \text{(A.10)}\end{aligned}$$

where the equality follows from the fact that $\zeta_1'/(1-\zeta_1) = \zeta_2'/(\zeta_2-1)$.

Note that the first term in the expression above is positive. Therefore, the sign of $\partial W(1)/\partial\lambda$ hinges on the sign of the second term. Since $\zeta_2 > 1 > \zeta_1 > 0$ and $\zeta_2' > 0$, we need only focus on the summation. Letting $k^* := \lfloor\zeta_2/(\zeta_2-1)\rfloor \geq 1$, we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k+1)\\ &\quad= \sum_{k=1}^{k^*}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k+1) + \sum_{k=k^*+1}^{\infty}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k+1)\\ &\quad> \sum_{k=1}^{k^*}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k^*+1) + \sum_{k=k^*+1}^{\infty}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\widehat{Y}(k^*+1)\\ &\quad= \widehat{Y}(k^*+1)\sum_{k=1}^{\infty}\left(\frac{\zeta_2}{\zeta_2-1}-k\right)\zeta_2^{-k}\\ &\quad= \widehat{Y}(k^*+1)\left(\frac{\zeta_2}{\zeta_2-1}\sum_{k=1}^{\infty}\zeta_2^{-k} + \sum_{k=1}^{\infty}k\zeta_2^{-k}\right) = \widehat{Y}(k^*+1)\left(\frac{\zeta_2}{\zeta_2-1}\frac{1}{\zeta_2-1} - \frac{\zeta_2}{(\zeta_2-1)^2}\right) = 0,\end{aligned}$$

where the inequality comes from the fact that $\widehat{Y}(k+1) > \widehat{Y}(k^*+1)$ for $k < k^*$ and $\widehat{Y}(k+1) < \widehat{Y}(k^*+1)$ for $k > k^*$. Thus, we have shown that $\partial W(1)/\partial\lambda > 0$. □

With this result in hand, we may move on to show that $\partial W(n)/\partial\lambda > 0$.

CLAIM. *For all* $n \geq 2$, $\partial W(n)/\partial\lambda > 0 > \partial\Pi(n)/\partial\lambda$.

PROOF OF CLAIM. Recall from Equation (7) that

$$W(n+1) = \frac{r+\lambda+\rho}{\rho}W(n) - \frac{\lambda}{\rho}\left(\widehat{Y}(n) + W(n-1)\right).$$

Differentiating with respect to λ yields

$$\frac{\partial W(n+1)}{\partial\lambda} = \frac{r+\lambda+\rho}{\rho}\frac{\partial W(n)}{\partial\lambda} - \frac{\lambda}{\rho}\frac{\partial W(n-1)}{\partial\lambda} - \frac{1}{\rho}\left(W(n-1) - W(n) + \widehat{Y}(n)\right).$$

Notice that this is a linear second-order (nonhomogeneous) difference equation (with $\partial W(n+1)/\partial\lambda$ a linear function of $\partial W(n)/\partial\lambda$ and $\partial W(n-1)/\partial\lambda$) with the boundary condition

$$\frac{\partial W(2)}{\partial\lambda} = \frac{\partial W(1)}{\partial\lambda}.$$

A solution for this system may be found in a manner analogous to that of Theorem 1. In particular, we have

$$\frac{\partial W(n)}{\partial\lambda} = \frac{\zeta_2^{n-1}}{\zeta_2-\zeta_1}\left((1-\zeta_1)\frac{\partial W(1)}{\partial\lambda} - \frac{1}{\rho}\sum_{k=1}^{n-1}\zeta_2^{-k}\left(W(k) - W(k+1) + \widehat{Y}(k+1)\right)\right)$$
$$+ \frac{\zeta_1^{n-1}}{\zeta_2-\zeta_1}\left((\zeta_2-1)\frac{\partial W(1)}{\partial\lambda} + \frac{1}{\rho}\sum_{k=1}^{n-1}\zeta_1^{-k}\left(W(k) - W(k+1) + \widehat{Y}(k+1)\right)\right).$$

Since $\widehat{Y}(k+1) > 0$ and $W(k) > W(k+1)$ for all $k \in \mathbb{N}$ (as demonstrated in Proposition 1), the second term above is always positive. Thus, the sign of $\partial W(n)/\partial\lambda$ will depend upon the sign of the first term. So, in order to reach a contradiction, suppose that

$$\rho(1-\zeta_1)\frac{\partial W(1)}{\partial\lambda} < \sum_{k=1}^{\infty}\zeta_2^{-k}\left(W(k) - W(k+1) + \widehat{Y}(k+1)\right).$$

Substituting in $\partial W(1)/\partial\lambda$ from Equation (A.10) and simplifying yields

$$\frac{\zeta_1\zeta_2}{\zeta_2-\zeta_1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) - \frac{\zeta_1(\zeta_2-1)}{\zeta_2-\zeta_1}\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1) < \sum_{k=1}^{\infty}\zeta_2^{-k}(W(k) - W(k+1)).$$

Recall from Equation (A.3) in the proof of Proposition 1 that

$$W(k) - W(k+1) = (1-\zeta_1)W(k) - \zeta_1\zeta_2^k\sum_{j=k}^{\infty}\zeta_2^{-j}\widehat{Y}(j+1).$$

Substituting this into the preceding expression and then multiplying by $(\zeta_2-\zeta_1)/(1-\zeta_1)$ yields

$$W(1) - \frac{\zeta_1(\zeta_2-1)}{1-\zeta_1}\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1) < (\zeta_2-\zeta_1)\sum_{k=1}^{\infty}\zeta_2^{-k}W(k) - \frac{\zeta_1(\zeta_2-\zeta_1)}{1-\zeta_1}\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1).$$

We may rearrange the above expression in order to arrive at

$$
\begin{aligned}
W(1) &< \zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k) - \zeta_1 \left(\sum_{k=1}^{\infty} \zeta_2^{-k} W(k) + \sum_{k=1}^{\infty} k \zeta_2^{-k} \widehat{Y}(k+1) \right) \\
&= \zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k) - \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) \\
&= W(1) + \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) - \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) = W(1),
\end{aligned}
$$

a contradiction, where in the second line we use the relationship in Equation (A.4). Therefore, the first term in the expression above for $\partial W(n)/\partial \lambda$ is positive, implying that the sum is also positive. Thus, $\partial W(n)/\partial \lambda > 0$. □

Finally, in order to show that $\partial W(n)/\partial \rho < 0$ for arbitrary n, we first need to show that this holds for $n = 1$.

CLAIM. $\partial W(1)/\partial \rho < 0$.

PROOF OF CLAIM. Recall from Equation (10) that

$$
W(1) = \frac{\zeta_1 \zeta_2}{1 - \zeta_1} \sum_{k=1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1).
$$

Let ζ_1' denote $\partial \zeta_1/\partial \rho$ and ζ_2' denote $\partial \zeta_2/\partial \rho$, and note that $\zeta_1' < 0$ and $\zeta_2' < 0$. We may write

$$
\begin{aligned}
\frac{\partial W(1)}{\partial \rho} &= \frac{\zeta_1 \zeta_2}{1 - \zeta_1} \sum_{k=1}^{\infty} -k \zeta_2^{-k-1} \zeta_2' \widehat{Y}(k+1) + \left(\frac{\zeta_1 \zeta_2' + \zeta_1' \zeta_2}{1 - \zeta_1} + \frac{\zeta_1 \zeta_2 \zeta_1'}{(1 - \zeta_1)^2} \right) \sum_{k=1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) \\
&= \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \left[\left(1 + \frac{\zeta_1 \zeta_2}{\zeta_1 \zeta_2'} + \frac{\zeta_1' \zeta_2}{(1 - \zeta_1) \zeta_2'} \right) \sum_{k=1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1) - \sum_{k=1}^{\infty} k \zeta_2^{-k} \widehat{Y}(k+1) \right] \\
&= \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \sum_{k=1}^{\infty} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \widehat{Y}(k+1). \qquad \text{(A.11)}
\end{aligned}
$$

Letting $k^* := \lfloor \zeta_2/(\zeta_2 - 1) \rfloor \geq 1$, we then have

$$
\begin{aligned}
\frac{\partial W(1)}{\partial \rho} &= \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \left(\sum_{k=1}^{k^*} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \widehat{Y}(k+1) + \sum_{k=k^*+1}^{\infty} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \widehat{Y}(k+1) \right) \\
&< \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \left(\sum_{k=1}^{k^*} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \widehat{Y}(k^*+1) + \sum_{k=k^*+1}^{\infty} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \widehat{Y}(k^*+1) \right) \\
&= \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \widehat{Y}(k^*+1) \sum_{k=1}^{\infty} \left(\frac{\zeta_2}{\zeta_2 - 1} - k \right) \zeta_2^{-k} \\
&= \frac{\zeta_1 \zeta_2'}{1 - \zeta_1} \widehat{Y}(k^*+1) \left(\frac{\zeta_2}{\zeta_2 - 1} \sum_{k=1}^{\infty} \zeta_2^{-k} + \sum_{k=1}^{\infty} k \zeta_2^{-k} \right) = 0,
\end{aligned}
$$

where the inequality comes from the fact that $\widehat{Y}(k+1) > \widehat{Y}(k^*+1)$ for $k < k^*$ and $\widehat{Y}(k+1) < \widehat{Y}(k^*+1)$ for $k > k^*$, as well as $\zeta_2' < 0$. Thus, we have shown that $\partial W(1)/\partial \rho < 0$. □

We may now finally proceed to show that $\partial W(n)/\partial\rho < 0 < \partial\Pi(n)/\partial\rho$.

CLAIM. *For all $n \geq 2$, $\partial W(n)/\partial\rho < 0$.*

PROOF OF CLAIM. Recall from Equation (7) that

$$W(n+1) = \frac{r+\lambda+\rho}{\rho}W(n) - \frac{\lambda}{\rho}\left(\widehat{Y}(n) + W(n-1)\right).$$

Differentiating with respect to ρ yields

$$\begin{aligned}\frac{\partial W(n+1)}{\partial\rho} &= \frac{r+\lambda+\rho}{\rho}\frac{\partial W(n)}{\partial\rho} - \frac{\lambda}{\rho}\frac{\partial W(n-1)}{\partial\rho} + \frac{\lambda}{\rho^2}\left(\widehat{Y}(n) + W(n-1) - \frac{r+\lambda}{\lambda}W(n)\right)\\ &= \frac{r+\lambda+\rho}{\rho}\frac{\partial W(n)}{\partial\rho} - \frac{\lambda}{\rho}\frac{\partial W(n-1)}{\partial\rho} + \frac{1}{\rho}\left(\frac{\lambda}{\rho}\widehat{Y}(n) + \frac{\lambda}{\rho}W(n-1) - \frac{r+\lambda}{\rho}W(n)\right)\\ &= \frac{r+\lambda+\rho}{\rho}\frac{\partial W(n)}{\partial\rho} - \frac{\lambda}{\rho}\frac{\partial W(n-1)}{\partial\rho} + \frac{1}{\rho}\left(W(n) - W(n+1)\right).\end{aligned}$$

Notice that this is a linear second-order (nonhomogeneous) difference equation (with $\partial W(n+1)/\partial\rho$ a linear function of $\partial W(n)/\partial\rho$ and $\partial W(n-1)/\partial\rho$) with the boundary condition

$$\frac{\partial W(2)}{\partial\rho} = \frac{\partial W(1)}{\partial\rho}.$$

A solution for this system may be found in a manner analogous to that of Theorem 1. In particular, we have

$$\begin{aligned}\frac{\partial W(n)}{\partial\rho} = &\frac{\zeta_2^{n-1}}{\zeta_2-\zeta_1}\left((1-\zeta_1)\frac{\partial W(1)}{\partial\rho} + \frac{1}{\rho}\sum_{k=1}^{n-1}\zeta_2^{-k}\left(W(k+1) - W(k+2)\right)\right)\\ &+ \frac{\zeta_1^{n-1}}{\zeta_2-\zeta_1}\left((\zeta_2-1)\frac{\partial W(1)}{\partial\rho} - \frac{1}{\rho}\sum_{k=1}^{n-1}\zeta_1^{-k}\left(W(k+1) - W(k+2)\right)\right).\end{aligned}$$

Since $W(k+1) > W(k+2)$ for all $k \in \mathbb{N}$ (as demonstrated in Proposition 1), the second term above is always negative. Thus, the sign of $\partial W(n)/\partial\rho$ will depend upon the sign of the first term. So, in order to reach a contradiction, suppose that

$$\sum_{k=1}^{\infty}\zeta_2^{-k}\left(W(k+1) - W(k+2)\right) > -\rho(1-\zeta_1)\frac{\partial W(1)}{\partial\rho}.$$

Substituting in $\partial W(1)/\partial\rho$ from Equation (A.11) and the expression for $W(k+1) - W(k+2)$ from Equation (A.3) and simplifying yields

$$\begin{aligned}(1-\zeta_1)\sum_{k=1}^{\infty}\zeta_2^{-k}W(k+1) - \zeta_1\zeta_2\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1)&\\ > \frac{\zeta_1^2\zeta_2}{\zeta_2-\zeta_1}\sum_{k=1}^{\infty}\zeta_2^{-k}\widehat{Y}(k+1) - \frac{\zeta_1\zeta_2(\zeta_2-1)}{\zeta_2-\zeta_1}\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1).&\end{aligned}$$

This implies that

$$(\zeta_2-\zeta_1)\sum_{k=1}^{\infty}\zeta_2^{-k}W(k+1) > \zeta_1 W(1) + \zeta_1\zeta_2\sum_{k=1}^{\infty}k\zeta_2^{-k}\widehat{Y}(k+1),$$

or, equivalently, that

$$\zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) - \zeta_1 \left(\sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) + \zeta_2 \sum_{k=1}^{\infty} (k-1)\zeta_2^{-k} \widehat{Y}(k+1) \right)$$
$$> \zeta_1 W(1) + \zeta_1 \zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} \widehat{Y}(k+1),$$

Note that, using Equation (A.4), we may write

$$\begin{aligned}\sum_{k=1}^{\infty} \zeta_2^{-k} W(k+2) &= \zeta_1 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) + \zeta_1 \zeta_2 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \zeta_2^{-j} \widehat{Y}(j+1) \\ &= \zeta_1 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) + \zeta_1 \zeta_2 \sum_{k=1}^{\infty} (k-1)\zeta_2^{-k} \widehat{Y}(k+1).\end{aligned}$$

Thus, the preceding inequality may be rewritten as

$$\zeta_2 \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+1) - \sum_{k=1}^{\infty} \zeta_2^{-k} W(k+2) > \zeta_1 W(1) + (1-\zeta_1) W(1),$$

or, equivalently,

$$W(2) > W(1).$$

This is a contradiction, however, as $W(2) = W(1)$. Therefore, the first term in the expression for $\partial W(n)/\partial \rho$ is negative, implying that $\partial W(n)/\partial \rho < 0$, as desired. □

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