Tighter Bounds for Facility Games*

Pinyan Lu †
Microsoft Research Asia

Yajun Wang ‡
Microsoft Research Asia

Yuan Zhou §
CMU

Abstract

In facility games, public facilities are placed based on the reported locations of the agents. The cost of an agent is measured by the distance from its location to the nearest facility. In this paper, we consider the one dimensional facility games where all locations for agents and the facilities are on the real line.

We study the approximation ratio of social welfare for *strategy-proof* mechanisms, where no agent can benefit by misreporting its location. The social welfare function studied in this paper is the total cost of the agents. We mainly study two extensions of the simplest version as in [10]: two facilities and multiple locations per agent. In both cases, no lower bound for randomized *strategy-proof* mechanisms was previously known. We prove the first lower bound of 1.045 for two-facility games and the first lower bound of 1.33 for multiple locations per agent setting. Our later lower bound is obtained by solving a related linear programming problem, and we believe that this new technique of proving lower bounds for randomized mechanisms may find applications in other problems and is of independent interest.

We also improve several approximation bounds in [10]. In particular, we give a tighter analysis of a randomized mechanism proposed by [10]. This analysis is quite involved and confirms a conjecture in [10]. We also give a simple randomized mechanism for the two-facility games with approximation ratio $n/2$, improving the naive $n-2$ ratio from the deterministic mechanisms. For deterministic mechanisms for two-facility games, we improve the approximation lower bound to 2 from 1.5.

1 Introduction

In a facility game, a planner is building public facilities while agents (players) are submitting their locations. In this paper, we study the facility game in one dimension, i.e., the locations of the agents and the facilities are in the real line. Let the position reported by agent i be $x_i \in \mathcal{R}_i \subseteq \mathcal{R}$. Assume the number of agents is n and the number of public facilities available is k. A (deterministic) mechanism for the k-facility game is simply a function

$$f : \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n \to \mathcal{R}^k.$$

In this paper, we assume $\mathcal{R}_i = \mathcal{R}$ for all agents. The *cost* of an agent is the distance from its *true* location to the nearest facility. Let $\{l_1, l_2, \ldots, l_k\}$ be the set of locations of the facilities. The cost of agent i is $\mathrm{cost}(\{l_1, \ldots, l_k\}, x_i) = \min_{1 \le j \le k} |x_i - l_j|$. A randomized mechanism returns a distribution over $\mathcal{R}^k$. Then the cost of agent i is the expected cost over the distribution returned by the randomized mechanism.

*Microsoft Research Technical Report

†pinyanl@microsoft.com
‡yajunw@microsoft.com
§yuanzhou@cmu.edu

An agent may misreport its location if it can reduce its own cost. A usual solution concept is *strategy-proofness*, which is also the focus of this paper. In a *strategy-proof* mechanism, no agent can unilaterally misreport its location to reduce its own cost. For $\mathbf{x} = \{x_1, x_2, \ldots, x_i, \ldots, x_n\} \in \mathcal{R}^n$, we define $\mathbf{x}_{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. A mechanism is *strategy-proof* if for any x_i and $x_i' \neq x_i$, $\text{cost}(f(\mathbf{x}_{-i}, x_i), x_i) \leq \text{cost}(f(\mathbf{x}_{-i}, x_i'), x_i)$. In other words, no matter what other agents' strategies are, one of the best strategies for agent i is reporting its true location.

The facility game problem has a rich history in social science literature. Consider the case that we are building one facility in a discrete set of locations (alternatives). Agents are reporting its preference for the alternatives. The renowned Gibbard-Satterthwaite theorem [6, 11] showed that if the preference on alternatives for each agent can be arbitrary, the only strategy-proof mechanisms are the dictatorships when the number of alternatives are greater than two.

In the facility game, however, the preferences on facility locations are not arbitrary. In particular, if the location of agent i is x_i, for locations $x < y < x_i$, it prefers y over x, while if $x_i < x < y$, it prefers x over y. In other words, agent i has a single preferred location x_i. When two locations are on the same side of x_i, agent i will always prefer the one closer to x_i. This kind of admissible individual preferences are defined as *single-peaked* preferences, which was first discussed by Black [3].

Since the Gibbard-Satterthwaite theorem does not hold in the setting for single-peaked preferences, the facility game admits a much richer set of strategy-proof mechanisms. For example, in case of an odd number of agents with one facility, the mechanism choosing the median location of the agents is a non-dictatorial strategy-proof mechanism.

Moulin [9] characterized the class of all strategy-proof mechanisms for *one facility* game in the real line. (One unnecessary assumption in the proof is dropped by Barberà and Jakson [2], and Sprumont [13].) In particular, a *generalized median voter scheme* is sufficient to characterize all strategy-proof mechanisms. Interested readers may refer to the detailed survey by Barberà [1] The characterization for the strategy-proof mechanisms with two or more facilities is widely open. Miyagawa [8] characterized the strategy-proof, Pareto-optimal, and continuous mechanisms for two facilities. This characterization, though interesting, only covers two simple mechanisms.

More recently, Procaccia and Tennenholtz [10] studied the facility game in a different perspective. They consider the facility game as a special case of the game theoretic optimization problems where the optimal social welfare solution is not strategy-proof. They treat the facility game in a broader concept of the games that payments are not allowed or infeasible. Such mechanism design problems without payments are rarely studied by computer scientists, except some special problems [12].

In a more algorithmic point of view, Procaccia and Tennenholtz studied strategy-proof mechanisms that give provable approximation ratios on social welfare, when the optimal solution is not strategy-proof. For the simplest case, only one facility is allocated, the median mechanism is both strategy-proof and optimal for social welcome. Then Procaccia and Tennenholtz studied two extensions: (1) there are two facilities to be located; (2) each agent controls multiple locations (one facility is to be located). In both cases, the optimal solutions are no longer strategy-proof in general. Therefore, we are interested in giving the best (from an approximate ratio point of view) strategy-proof mechanisms for these extensions. This is also the focus of this paper. A strategy-proof mechanism has an approximation ratio of α if for every input instance, the social cost for the output of the mechanism is always at most α times the social cost for an optimal solution.

We remark that, if payment is allowed, then the well-know Vickrey-Clarke-Groves (VCG) mechanism [14, 4, 7] will give both optimal and strategy-proof solutions for both extensions. However, in many real scenarios, payment is not available as noted by Schummer and Vohra [12]. We focus on the strategy-proof mechanisms without money in this paper.

1.1 Our result

We study the approximation ratios of social welfare for the strategy-proof mechanisms in the facility game with one or more facilities. The social welfare function we use is the *social cost*, i.e., the total cost of all agents. We mainly focus on the approximation ratios for *social cost* of the strategy-proof mechanisms, where we improve most results in [10]. Furthermore, we also provide several novel approximation bounds which are not previously available. Table 1 summarizes our contribution.

	Two Facilities	Multi-Location Per Agent (One Facility)
Deterministic	UB: $(n-2)$ LB: **2**(1.5)	UB: (3 [5]) LB: (3 [5])
Randomized	UB: $\mathbf{n/2}$ $(n-2)$ LB: **1.045**(N/A)	UB: $\mathbf{3 - \frac{2\min_{j\in N} w_j}{\sum_{j\in N} w_j}}$ $(2+\frac{\lvert w_1-w_2\rvert}{w_1+w_2}$ for $n=2$ only) LB: **1.33** (N/A)

Table 1: Our results are in bold. The numbers in brackets are previous results in [10] unless stated otherwise. (N/A means no previous known bound.)

The organization of the paper is as follows. In Section 2, we provide improved upper and lower bounds of both deterministic and randomized strategy-proof mechanisms for two-facility games. In Section 3, we study the cases when each agent controls more than one location. We conclude our paper in Section 4 with several open problems.

2 Two-Facility Games

In this section, we study strategy-proof mechanisms for two-facility games. We first provide a better randomized mechanism achieving approximation ratio of $n/2$. The only previous know upper bound is $n-2$, which is from deterministic mechanisms. Then we study the lower bounds both for deterministic and randomized cases. For deterministic mechanisms, the lower bound is improved to 2 from 1.5 in [10]. For randomized mechanisms, we provide the first non-trivial approximation ratio lower bound of 1.045.

2.1 A Better Randomized Mechanism

The following mechanism is inspirited by Mechanism 2 from [10]. However, our proof is different and much simpler than theirs.

Mechanism 1. See Figure 1 for reference. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be the reported locations of the agents. Define $\mathrm{lt}(\mathbf{x}) = \min\{x_i\}$, $\mathrm{rt}(\mathbf{x}) = \max\{x_i\}$ and $\mathrm{mt}(\mathbf{x}) = (\mathrm{lt}(\mathbf{x}) + \mathrm{rt}(\mathbf{x}))/2$. We further define the *left boundary* $\mathrm{lb}(\mathbf{x}) = \max\{x_i : i \in N, x_i \leq \mathrm{mt}(\mathbf{x})\}$ and the *right boundary* $\mathrm{rb}(\mathbf{x}) = \min\{x_i : i \in N, x_i \geq \mathrm{mt}(\mathbf{x})\}$. Let $\mathrm{dist}(\mathbf{x}) = \max\{\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x}), \mathrm{lb}(\mathbf{x}) - \mathrm{lt}(\mathbf{x})\}$. We set $\overline{\mathrm{lb}}(\mathbf{x}) = \mathrm{lt}(\mathbf{x}) + \mathrm{dist}(\mathbf{x})$ and $\overline{\mathrm{rb}}(\mathbf{x}) = \mathrm{rt}(\mathbf{x}) - \mathrm{dist}(\mathbf{x})$. The mechanism return $(\mathrm{lt}(\mathbf{x}), \mathrm{rt}(\mathbf{x}))$ or $(\overline{\mathrm{lb}}(\mathbf{x}), \overline{\mathrm{rb}}(\mathbf{x}))$, both with probability 1/2.

Theorem 2.1. *Mechanism 1 is strategy-proof. And the approximation ratio of Mechanism 1 is $n/2$ for social cost.*

Proof. We first prove the approximation ratio assuming that all agents report their true locations. By symmetry, we assume $\mathrm{rt}(\mathbf{x}) - \mathrm{rb}(\mathbf{x}) \leq \mathrm{lb}(\mathbf{x}) - \mathrm{lt}(\mathbf{x})$ as in Figure 1. Since we only have two facilities, either $\mathrm{lt}(\mathbf{x})$ and $\mathrm{rb}(\mathbf{x})$ or $\mathrm{rb}(\mathbf{x})$ and $\mathrm{rt}(\mathbf{x})$ are served by a same facility. Therefore the optimal solution

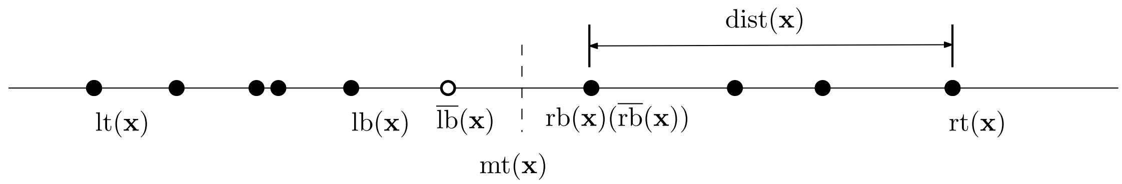

Figure 1: Mechanism 1 pick $(\text{lt}(\mathbf{x}), \text{rt}(\mathbf{x}))$ or $(\overline{\text{lb}}(\mathbf{x}), \overline{\text{rb}}(\mathbf{x}))$ both with probability 1/2

is least $\min\{|\text{lt}(\mathbf{x}) - \text{rb}(\mathbf{x})|, |\text{rb}(\mathbf{x}) - \text{rt}(\mathbf{x})|\} = \text{dist}(\mathbf{x})$. On the other hand, for each agent, its expected cost is exactly $\text{dist}(\mathbf{x})/2$ in this mechanism. So Mechanism 1 is $\frac{n}{2}$-approximate.

We then show that Mechanism 1 is strategy-proof. We first show that any point other than the 3 points defining $\text{lt}(\mathbf{x}), \text{rt}(\mathbf{x})$ and $\text{rb}(\mathbf{x})$ cannot benefit by misreporting its location. Let the new configuration is $\mathbf{x}'$. Consider the 3 points defining the previous $\text{lt}(\mathbf{x}), \text{rt}(\mathbf{x})$ and $\text{rb}(\mathbf{x})$. No matter how the 3 points are partitioned by the new $\text{mt}(\mathbf{x}')$, $\text{dist}(\mathbf{x}') \geq \text{rt}(\mathbf{x}) - \text{rb}(\mathbf{x})$, where $\mathbf{x}'$ is the new configuration. We know that the expected cost for any location in this configuration is at lease $\text{dist}(\mathbf{x}')/2$, which is at least as large as the honest cost $\text{dist}(\mathbf{x}) = \text{rt}(\mathbf{x}) - \text{rb}(\mathbf{x})$. The same argument also shows $\text{lt}(\mathbf{x})$ (resp. $\text{rt}(\mathbf{x})$) does not have incentive of reporting positions on the left (resp. right).

Consider the point $\text{rb}(\mathbf{x})$. Its expected cost is $\frac{\text{rt}(\mathbf{x})-\text{rb}(\mathbf{x})}{2}$ if it reports its true location. By lying, it cannot move the left or right boundary towards itself, and as a result, its expected cost in any new configuration is at least $\frac{\min\{|\text{lt}(\mathbf{x})-\text{rb}(\mathbf{x})|,|\text{rb}(\mathbf{x})-\text{rt}(\mathbf{x})|\}}{2} = \frac{\text{rt}(\mathbf{x})-\text{rb}(\mathbf{x})}{2}$. Therefore, the point at $\text{rb}(\mathbf{x})$ has no incentive to lie.

The only possible cases left to analyze is that the agent at $\text{lt}(\mathbf{x})$ (resp. $\text{rt}(\mathbf{x})$) reporting a location to the right (resp. left). Its expected cost is $\frac{\text{rt}(\mathbf{x})-\overline{\text{lb}}(\mathbf{x})}{2}$ if it reports its true location. Reporting a location on its right can only move $\overline{\text{lb}}(\mathbf{x}')$ toward right, which will only hurt itself. Therefor the agent at $\text{lt}(\mathbf{x})$ has no incentive to lie. Similar argument also holds if the agent at $\text{rt}(\mathbf{x})$ reports its location on the left of $\text{rt}(\mathbf{x})$.

To sum up, no agent has incentive to lie. Therefore **Mechanism 1** is strategy-proof. □

2.2 Lower Bounds

In this subsection, we prove improved lower bounds both for deterministic and randomized strategy-proof mechanisms. Both bounds are proved by the following construction, which is similar to the 1.5 lower bound example in [10].

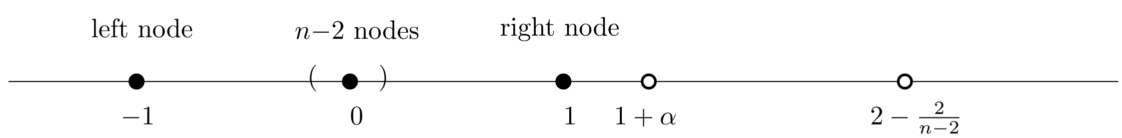

Figure 2: Lower bound example for two-facility game.

Theorem 2.2 (Lower bound for deterministic mechanisms.)**.** *In a two-facility game, any deterministic strategy-proof mechanism $f : \mathcal{R}^n \to \mathcal{R}^2$ has an approximation ratio of at least $2 - \frac{4}{n-2}$ for the social cost.*

Proof. See Figure 2 for the configuration. We have $n-2$ nodes at the origin and the left node at -1 and the right node at 1.

Assume to the contrary, there exists a strategy-proof mechanism with approximation ratio less than 2. Then this mechanism has to place one facility in the range $(-\frac{2}{n-2}, \frac{2}{n-2})$. Now consider the left node and the right node at -1 and 1. At least one of them is $1 - 2/(n-2)$ away from its closest facility. Without loss of generality, assume the right node at 1 is at least $1 - \frac{2}{n-2}$ away from the facilities.

If there is one facility on the right of 1, it must be placed at a position right to $2 - 2/(n-2)$ by our assumption. In this case, since the optimal cost is 1, the approximation ratio is at least $2 - \frac{4}{n-2}$ as one facility is always close to the origin.

Now consider the case that the closest facility to the right node at 1 is on the left. Consider the image set I of the closest facility to the right node when it moves while all other nodes remain fixed. Clearly, by strategy-proofness, $I \cap (\frac{2}{n-2}, 2 - \frac{2}{n-2}) = \emptyset$. On the other hand, $I \cap [2 - \frac{2}{n-2}, +\infty) \neq \emptyset$, otherwise the approximation ratio is unbounded when the right node moves to the infinity.

Take p as the left most point of $I \cap [2 - \frac{2}{n-2}, +\infty)$. ($p$ always exists, as I is a closed set.) If we place the right node at $p - 1 + \frac{2}{n-2}$, the closest facility to x is at p. Therefore, the cost of the mechanism for such a configuration is at least $2 - \frac{4}{n-2}$, as the other facility has to be close to the origin. Because the optimal cost is still 1, the approximation ratio is at least $2 - \frac{4}{n-2}$. □

If the mechanism is randomized, the output is a distribution of $\mathcal{R}^2$. Notice that in a randomized mechanism, the cost of an agent is measured by the *expected distance* from its true location to the closest facility. We give the first non-trivial (greater than 1) approximation ratio lower bound of strategy-proof mechanisms for social cost in Theorem 2.3.

Theorem 2.3 (Lower bound for randomized mechanisms.)**.** *In a two facility game, any randomized strategy-proof mechanism has an approximation ratio of at least* $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{n-2} \geq 1.045 - \frac{1}{n-2}$ *for the social cost for any* $n \geq 5$.

Proof. Again, we consider the point set as in Figure 2. Let the expected distance from -1, 0 and 1 to the closest facility be e_1, e_2 and e_3 respectively. Clearly, we have $e_1 + e_2 + e_3 \geq 1$. For any randomized strategy-proof mechanism with approximation ratio at most 2, $e_2 \leq \frac{2}{n-2}$. Without loss of generality, we assume $e_3 \geq \frac{1}{2} - \frac{1}{n-2}$.

Now we place the right node at 1 to a new position at $1 + \alpha$ for some $\alpha \in (0, 1/2)$. Let e_3' be the expected distance from $1 + \alpha$ to the nearest facility at the new configuration by the same strategy-proof mechanism. Because of strategy-proofness, $e_3' \geq \frac{1}{2} - \alpha - \frac{1}{n-2}$. (The condition $n \geq 5$ guarantees $e_3' \geq 0$ for the optimal α chosen later.)

Let $p(x)$ be the probability density function of the probability the closest facility to the right node at $1 + \alpha$ is at x in the new configuration. When $x \leq -\frac{1}{n-2}$, the closest facility is at *weighted distance* at least 1 to nodes at 0. When $x \geq \frac{1}{n-2}$, for any placement of the other facility, the sum of the *weighted* distances to the closest facility for the nodes at -1 and 0 is at least 1. In these two cases, the weighted distance to nodes at -1 and 0 is at least 1. Denote $P = \int_{-\frac{1}{n-2}}^{\frac{1}{n-2}} p(x)\,\mathrm{d}x$. Therefore, the total cost of the mechanism in the new configuration is at least:

$$cost \geq (1 - P) \cdot 1 + e_3' \geq 1 + \frac{1}{2} - \alpha - \frac{1}{n-2} - P.$$

On the other hand, consider the distance to the node at $1 + \alpha$. When the closest facility to $1 + \alpha$ is $x \in (-\frac{1}{n-2}, \frac{1}{n-2})$, the total weighted distance from the nodes to the closest facilities is at least $1 + \alpha$. Therefore, we have

$$cost \geq (1 - P) \cdot 1 + P \cdot (1 + \alpha) = 1 + \alpha \cdot P.$$

Combining the two inequalities, the optimal ratio is achieved when $P = \frac{1/2-\alpha-1/(n-2)}{1+\alpha}$ and the approximation ratio is at least

$$1 + \frac{1}{2} - \alpha - \frac{1}{n-2} - \frac{1/2 - \alpha - 1/(n-2)}{1+\alpha} \geq 1 + \frac{1}{2} - \frac{1}{n-2} - \frac{\alpha^2 + 1/2}{1+\alpha}.$$

Define $g(\alpha) = \frac{\alpha^2+1/2}{1+\alpha}$. The maximum ratio is achieved when $g'(\alpha) = 0$ with $\alpha = \frac{2-\sqrt{2}}{4}$, and the approximation ratio is at least $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{n-2}$. □

Both lower bounds for deterministic and randomized strategy-proof mechanisms can be generalized to k facilities for $k \geq 3$. (Consider the configuration that two nodes on the two sides, and $k-1$ group of nodes in between. Each group of nodes (including the two singletons) are at unit distance away.) We have a direct corollary.

Corollary 2.1 (Lower bound for k-facility game.)**.** *In a k-facility game for $k \geq 2$, any deterministic strategy-proof mechanism has an approximation ratio of at least $2 - \frac{4}{m}$ for the social cost, where $m = \lfloor \frac{n-2}{k-1} \rfloor$. Any* randomized strategy-proof mechanism *for k-facility game has an approximation ratio of at least* $1 + \frac{\sqrt{2}-1}{12-2\sqrt{2}} - \frac{1}{m} \geq 1.045 - \frac{1}{m}$.

3 Multiple Locations Per Agent

In this section, we study the case that each agent controls multiple locations. Assume agent i controls w_i locations, i.e., $\mathbf{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{iw_i}\}$. A (deterministic) mechanism with one facility in the multiple locations setting is a function $f : \mathcal{R}^{w_1} \times \cdots \times \mathcal{R}^{w_n} \to \mathcal{R}$ for n agents. Then, for agent i, its cost is defined as $\text{cost}(l, \mathbf{x}_i) = \sum_{j=1}^{w_i} |l - x_{ij}|$, where l is the location of the facility. As before, we are interested in minimizing the social cost of the agents, i.e., $\sum_{i \in N} \sum_{j=1}^{w_i} |l - x_{ij}|$, where $N = \{1, 2, \ldots, n\}$.

We first give a tight analysis of a randomized truthful mechanism proposed in [10]. This in particular confirms a conjecture of [10]. Then we prove the first approximation ratio lower bound of 1.33 for any randomized mechanisms. This lower bound even holds for the simplest case of two player and each controls the same number of locations.

3.1 A Tight Analysis of a Randomized Mechanism

In [10], Procaccia and Tennenholtz proposed the following randomized mechanism in the multiple locations setting:

Randomized Median Mechanism: Given $\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, return $\text{med}(\mathbf{x}_i)$ with probability $w_i / (\sum_{j \in N} w_j)$.

If w_i is even, $\text{med}(\mathbf{x}_i)$ will report either the $\frac{w_i}{2}$th location or $\frac{w_i}{2}$+1st location of $\mathbf{x}_i$. In [10], Procaccia and Tennenholtz gave a tight analysis for the case of two players($n = 2$), which has an approximation ratio of $2 + \frac{|w_1 - w_2|}{w_1 + w_2}$. They proposed as an open question for the bound in the general setting. In this section, we give a tight analysis of this randomized mechanism in the general setting, which in particular confirms the conjecture. Notice that $2 + \frac{|w_1 - w_2|}{w_1 + w_2} = 3 - \frac{2 \min_{j \in N} w_j}{\sum_{j \in N} w_j}$, when $n = 2$.

Theorem 3.1. *The* Randomized Median Mechanism *has an approximate ratio of* $3 - \frac{2 \min_{j \in N} w_j}{\sum_{j \in N} w_j}$ *for social cost.*

Proof. If $n = 1$, then med (x_1) is the optimal solution. So the mechanism has an approximate ratio of $3 - \frac{2w_1}{w_1} = 1$. Now we consider the case for $n \geq 2$.

Without loss of generality, we can reorder the players so that $\text{med}(\mathbf{x}_1) \leq \text{med}(\mathbf{x}_2) \leq \cdots \leq \text{med}(\mathbf{x}_n)$. Then it must be the case that $\text{med}(\mathbf{x}_1) \leq \text{med}(\mathbf{x}) \leq \text{med}(\mathbf{x}_n)$. The idea here is to construct a worst case instance for this mechanism and then analyze the approximate ratio for the worst case. Let i' be the largest i such that $\text{med}(\mathbf{x}_i) \leq \text{med}(\mathbf{x})$.

Claim 1. *We can assume that the worst case satisfies the following properties: (1) w_i is even for all $i \in N$; (2) for all $i \leq i'$, $\text{med}(\mathbf{x}_i)$ returns the $\frac{w_i}{2}$-th point of $\mathbf{x}_i$; (3) and for all $i > i'$, $\text{med}(\mathbf{x}_i)$ returns the $(\frac{w_i}{2}+1)$-th point of $\mathbf{x}_i$.*

We justify the claim as following: if some w_i is odd, we can add one more point for agent i at the global median $\text{med}(\mathbf{x})$, then the original $\text{med}(\mathbf{x}_i)$ is still one of i-th two medians after adding the new point. We still return that value when we need to return $\text{med}(\mathbf{x}_i)$. After the modification, the expected cost can only increase while the optimal cost remain the same. So we can assume all w_i are even in a worst case. The properties (2) and (3) are obvious because returning the other point only improves the performance of the mechanism.

Now we assume that our instance satisfies all properties in Claim 1. By symmetry, we can further assume $\sum_{i=1}^{i'} w_i \geq \sum_{i=i'+1}^{n} w_i$. Let $W = \sum_{j \in N} w_j$ and $R(\text{med}(\mathbf{x}_i))$ be the rank of $\text{med}(\mathbf{x}_i)$ in the whole set $\mathbf{x}$. Let X be the ordered global set of $\mathbf{x}$ and X_i be the ith location in X. We perturb the points so that X_i and $R(\text{med}(\mathbf{x}_i))$ are well defined. Then for all $i \leq i'$, $R(\text{med}(\mathbf{x}_i)) \geq \sum_{j=1}^{i} \frac{w_i}{2}$; and for all $i > i'$, $R(\text{med}(\mathbf{x}_i)) \leq W - \sum_{j=i}^{n} \frac{w_i}{2}$. The worst case happens when the above two sets of inequalities all reach equalities.

We further make the two sides more symmetric as follows. If $w_1 > w_n$, previously, the mechanism returns $x_{\frac{w_1}{2}}$ with probability $\frac{w_1}{W}$ and returns $X_{W+1-\frac{w_n}{2}}$ with probability $\frac{w_n}{W}$. We modify the mechanism by returning $X_{\frac{w_n}{2}}$ and $X_{W+1-\frac{w_n}{2}}$ both with probability $\frac{w_n}{W}$ and returning $x_{\frac{w_1}{2}}$ with probability $\frac{w_1-w_n}{W}$. We continue this process and finally we can get the following mechanism. There are $0 = k_0 < k_1 < k_2 < \cdots < k_m$ and $l \leq m$. The mechanism returns X_{k_i} and X_{W+1-k_i} both with probability $\frac{k_i-k_{i-1}}{k_m+k_l}$ if $1 \leq i \leq l$; returns x_{k_i} with probability $\frac{k_i-k_{i-1}}{k_m+k_l}$ if $l < i \leq m$. (The meaning of k_js are roughly $k_j = \sum_{i=1}^{j} \frac{w_i}{2}$. However due to the symmetrization process descried above, we also have $k_j = \sum_{i=n}^{n-j} \frac{w_i}{2}$ for $j \leq l$.) We have $k_1 = \frac{\min\{w_1, w_n\}}{2} \geq \frac{\min_{j \in N} w_j}{2}$ and $k_m + k_l = W/2$.

The optimal solution is $\text{OPT} = \sum_{i=1}^{W/2}(X_{W+1-i} - X_i)$. Let $a_j = \sum_{i=k_{j-1}+1}^{k_j}(X_{W+1-i} - X_i)$. Then $\text{OPT} \geq s = \sum_{j=1}^{m} a_j$. Now we can compute the expected cost for this mechanism. For $1 \leq i \leq l$, we calculate the cost for X_{k_i} and X_{W+1-k_i} together. They both have probability $\frac{k_i-k_{i-1}}{k_m+k_l}$. The cost for X_{k_i}

is

$$\begin{aligned}
& \sum_{j=1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|) \\
= & \sum_{j=1}^{k_i}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|) + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|) \\
= & \sum_{j=1}^{k_i}|X_{W+1-j} - X_j| + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|) \\
= & \sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|)
\end{aligned}$$

Similarly, we can show that the cost for X_{W+1-k_i} is

$$\sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{W+1-k_i}| + |X_{W+1-j} - X_{W+1-k_i}|)$$

Combing the cost of X_{k_i} and X_{n+1-k_i} together, we have

$$\begin{aligned}
& 2\sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}| + |X_j - X_{W+1-k_i}| + |X_{W+1-j} - X_{W+1-k_i}|) \\
= & 2\sum_{j=1}^{i} a_j + 2\sum_{j=k_i+1}^{W/2}|X_{W+1-k_i} - X_{k_i}| \\
= & 2\sum_{j=1}^{i} a_j + 2(k_m + k_l - k_i)|X_{W+1-k_i} - X_{k_i}| \\
\leq & 2\sum_{j=1}^{i} a_j + 2(k_m + k_l - k_i)\frac{a_i}{k_j - k_{j-1}}
\end{aligned}$$

Now consider the case for $l + 1 \leq i \leq m$. The cost of X_{k_i} is

$$\begin{aligned}
& \sum_{j=1}^{i} a_j + \sum_{j=k_i+1}^{W/2}(|X_j - X_{k_i}| + |X_{W+1-j} - X_{k_i}|) \\
\leq & \sum_{j=1}^{i} a_j + 2(k_m + k_l - k_i)|X_{W+1-k_i} - X_{k_i}| \leq \sum_{j=1}^{i} a_j + 2(k_m + k_l - k_i)\frac{a_i}{k_j - k_{j-1}}
\end{aligned}$$

We are ready to derive the expected cost of the mechanism as follows.

$$
\begin{aligned}
&\text{Expected Cost} \\
\leq\; & \sum_{j=1}^{l} \frac{k_j - k_{j-1}}{k_m + k_l}(2\sum_{i=1}^{j} a_i + 2\frac{a_j(k_m + k_l - k_j)}{k_j - k_{j-1}}) + \sum_{j=l+1}^{m} \frac{k_j - k_{j-1}}{k_m + k_l}(\sum_{i=1}^{j} a_i + 2\frac{a_j(k_m + k_l - k_j)}{k_j - k_{j-1}}) \\
=\; & 2\sum_{j=1}^{l} \frac{k_j - k_{j-1}}{k_m + k_l} \sum_{i=1}^{j} a_i + \sum_{j=l+1}^{m} \frac{k_j - k_{j-1}}{k_m + k_l} \sum_{i=1}^{j} a_i + \frac{2}{k_m + k_l} \sum_{j=1}^{m} a_j(k_m + k_l - k_j) \\
=\; & \frac{1}{k_m + k_l}(2\sum_{i=1}^{l} a_i \sum_{j=i}^{l}(k_j - k_{j-1}) + \sum_{i=1}^{l} a_i \sum_{j=l+1}^{m}(k_j - k_{j-1}) + \sum_{i=l+1}^{m} a_i \sum_{j=i}^{m}(k_j - k_{j-1}) - 2\sum_{j=1}^{m} a_j k_j) + 2s \\
=\; & \frac{1}{k_m + k_l}(2\sum_{i=1}^{l} a_i(k_l - k_{i-1}) + \sum_{i=1}^{l} a_i(k_m - k_l) + \sum_{i=l+1}^{m} a_i(k_m - k_{i-1}) - 2\sum_{j=1}^{m} a_j k_j) + 2s \\
\leq\; & \frac{1}{k_m + k_l}(2k_l\sum_{i=1}^{l} a_i + (k_m - k_l)\sum_{i=1}^{l} a_i + k_m\sum_{i=l+1}^{m} a_i - 2k_1\sum_{j=1}^{m} a_j) + 2s \\
=\; & \frac{1}{k_m + k_l}((k_m + k_l)\sum_{i=1}^{l} a_i + k_m\sum_{i=l+1}^{m} a_i - 2k_1\sum_{j=1}^{m} a_j) + 2s \\
\leq\; & (3 - \frac{2k_1}{k_m + k_l})s \leq (3 - \frac{2\min_{j\in N} w_j}{\sum_{j\in N} w_j})\text{OPT}
\end{aligned}
$$

□

The following corollary confirms a conjecture of [10] regarding the case where each agent controls the same number of locations.

Corollary 3.1. *If all the players control the same number of locations, the approximate ratio of* Randomized Median Mechanism *is* $3 - \frac{2}{n}$.

3.2 Approximation Lower Bounds for Randomized Strategy-proof Mechanisms

In this section, we consider the lower bound of the approximation ratios for randomized strategy-proof mechanisms in the multiple locations setting. We first give a 1.2 lower bound of the approximation ratio, based on a very simple instance. Then we extend to a more complicated instance, which we derive a lower bound of 1.33 by solving a linear programming instance.

Theorem 3.2. *Any randomized strategy-proof mechanism of the one-facility game has an approximation ratio at least* 1.2 *in the setting that each agent controls multiple locations.*

Proof. We assume to the contrary that there exists one strategy-proof mechanism M which has an approximate ratio $c < 1.2$. Consider the following three instances:

Instance 1 First player has 2 points on 0 and 1 points on 1; and second player has all 3 points on 1.

Instance 2 First player has all 3 points on 0; and second player has all 3 points on 1.

Instance 3 First player has all 3 points on 0; second player has 1 points on 0 and 2 points on 1;.

Let P_1, P_2 and P_3 be the distribution of the facility the mechanism M gives for these three instance respectively. For all $x \in R$ and a distribution P on R, we use $cost(P, x)$ to denote $E_{y \sim P}|y - x|$. Then we have (for all $i = 1, 2, 3$)

$$cost(P_i, 0) + cost(P_i, 1) \geq 1.$$

We use $p_1(x), p_2(x)$ and $p_3(x)$ to denote the probability density function of P_1, P_2 and P_3 respectively. Let

$$\forall i \in \{1, 2, 3\},\ L_i = \int_{-\infty}^{0} -x p_i(x) \mathrm{d}x \quad \text{and} \quad R_i = \int_{1}^{+\infty} (x - 1) p_i(x) \mathrm{d}x.$$

Now, we computer the cost of the players in each distribution. For the first player in Instance 1, its cost in distribution P_i is

$$\begin{aligned}
& 2cost(P_i, 0) + cost(P_i, 1) \\
= \;& cost(P_i, 0) + (cost(P_i, 0) + cost(P_i, 1)) \\
= \;& cost(P_i, 0) + \int_{-\infty}^{+\infty} (|x| + |x - 1|) p_i(x) dx \\
= \;& cost(P_i, 0) + (\int_{-\infty}^{0} (1 - 2x) p_i(x) dx + \int_{0}^{1} p_i(x) dx + \int_{1}^{+\infty} (2x - 1) p_i(x) dx) \\
= \;& cost(P_i, 0) + (2 \int_{-\infty}^{0} (-x) p_i(x) dx + \int_{-\infty}^{+\infty} p_i(x) dx + 2 \int_{1}^{+\infty} (x - 1) p_i(x) dx) \\
= \;& cost(P_i, 0) + 2L_i + 2R_i + 1
\end{aligned}$$

So the strategy-proofness (of the first player in Instance 1) requires the following inequality:

$$cost(P_1, 0) + 2(L_1 + R_1) \leq cost(P_2, 0) + 2(L_2 + R_2).$$

Since $L_1, R_1 \geq 0$, we have

$$cost(P_1, 0) \leq cost(P_2, 0) + 2(L_2 + R_2). \tag{1}$$

By symmetry, we also have

$$cost(P_3, 1) \leq cost(P_2, 1) + 2(L_2 + R_2). \tag{2}$$

Using the similar calculation as above, we can get the expected cost of Instance 1 as follows.

$$2cost(P_1, 0) + 4cost(P_1, 1) = 2cost(P_1, 1) + 2(2L_1 + 1 + 2R_1) \geq 2cost(P_1, 1) + 2.$$

Since the optimal cost is 2 and the approximate ratio is less than 1.2, we have

$$2cost(P_1, 1) + 2 < 2 \times 1.2 = 2.4.$$

Therefore, we have $cost(P_1, 1) < 0.2$ and hence $cost(P_1, 0) > 0.8$. Substituting the above inequality into (1), we get

$$cost(P_2, 0) + 2(L_2 + R_2) > 0.8.$$

Again by symmetry, we also have

$$cost(P_2, 1) + 2(L_2 + R_2) > 0.8.$$

Adding these two inequalities, we have

$$cost(P_2, 0) + cost(P_2, 1) + 4(L_2 + R_2) > 1.6.$$

We also have

$$cost(P_2, 0) + cost(P_2, 1) = 1 + 2(L_2 + R_2).$$

Substituting this, we get

$$L_2 + R_2 > 0.1.$$

On the other hand, the approximate ratio condition of Instance 2 requires that

$$1 + 2(L_2 + R_2) < 1.2.$$

This is a contradiction. □

To prove the lower bound of 1.33, we extend the above instances as follows. We employ $2K + 1$ ($K \geq 1$ is an integer) instances (for $K = 1$, this is exactly the same set of instances as above):

Instance i ($1 \leq i \leq K$): First player has $K + i$ points on 0 and $K + 1 - i$ points on 1; second player has all $2K + 1$ points on 1.

Instance $K + 1$: First player has all $2K + 1$ points on 0; second player has all $2K + 1$ points on 1.

Instance i ($K + 2 \leq i \leq 2K + 1$): First player has all $2K+1$ points on 0; second player has $i - K - 1$ points on 0 and $3K + 2 - i$ points on 1.

Again, let P_i be the distribution of output of the mechanism on Instance i. Define the variables as $X_i = cost(P_i, 0)$ and $Y_i = cost(P_i, 1)$. Then, the strategy-proofness among the instances can be listed as linear inequalities (constrains). Assume the approximation ratio is α. The approximation ratio can also be bounded by linear constrains. In particular, we want to compute the minimal ratio α so that all constrains are satisfied. It is then straightforward to formulate the following linear programming problem.

$$\begin{aligned}
&\text{Minimize:} \quad \alpha \\
&\text{Subject to:} \\
&(K+i)X_i + (3K+2-i)Y_i \leq (K+i)\alpha, && i = 1, 2, \cdots, K+1 \\
&(K+i)X_i + (3K+2-i)Y_i \leq (3K+2-i)\alpha, && i = K+2, K+3, \cdots, 2K+1 \\
&(K+i)X_i + (K+1-i)Y_i \leq (K+i)X_{i+1} + (K+1-i)Y_{i+1}, && i = 1, 2, \cdots, K \\
&(i-K-1)X_i + (3K+2-i)Y_i \leq (i-K-1)X_{i-1} + (3K+2-i)Y_{i-1}, && i = K+2, K+3 \cdots, 2K+1 \\
&X_i + Y_i \geq 1, && i = 1, 2, \cdots, 2K+1 \\
&X_i, Y_i \geq 0, && i = 1, 2, \cdots, 2K+1
\end{aligned}$$

First two sets of constrains come from the approximate ratio constrain. The next two sets of constrains are enforced by strategy-proofness. And the last two sets of constrains are boundary conditions.

Choosing $K = 500$, we can solve this LP problem by computer and the optimal value is greater than 1.33. Therefore, if we set the approximation ratio to 1.33, there is no feasible solution for the linear programming which implies no feasible strategy-proof mechanism for the instances. So we have an approximation lower bound of 1.33.

Theorem 3.3. *Any randomized strategy-proof mechanism of the one-facility game has an approximation ratio at least* 1.33 *in the setting that each agent controls multiple locations.*

The numerical computation suggests that the optimal value for this LP problem is close to $\frac{4}{3}$ when K is large. It would be interesting to give an analytical proof for a lower bound of $\frac{4}{3}$. We leave it as an open question.

Conjecture 1. *Any randomized strategy-proof mechanism of the one-facility game has an approximation ratio at least* $\frac{4}{3} - \epsilon$ *for any* $\epsilon > 0$ *in the setting that each agent controls multiple locations.*

4 Conclusion

In this paper, we study the strategy-proof mechanisms in facility games. In particular, we derive approximation bounds for such mechanisms for social cost both in the two-facility game setting and the multiple location setting. Our results improves several bounds previously studied [10]. We also obtain some new approximation lower bounds which were not available before.

There are still a lot of interesting open questions. For example, in the two-facility game, we only have a deterministic mechanism with approximation ratio of $n-2$ for social cost, while the lower bound is only 2. There is also a huge gap for the randomized strategy-proof mechanisms in the two-facility game, with current upper bound $n/2$ and a lower bound of 1.045.

One more interesting research problem is providing a characterization for the class of strategy-proof mechanisms for the two-facility games. The characterization for one-facility game has very elegant mathematical structure. It would be interesting to give a similar characterization for the multiple facility setting, even with certain assumptions (e.g., anonymity).

References

[1] S. Barberà. An introduction to strategy-proof social choice functions. *Social Choice and Welfare*, 18(4):619–653, 2001.

[2] S. Barberà and M. Jackson. A characterization of strategy-proof social choice functions for economies with pure public goods. *Social Choice and Welfare*, 11(3):241–252, 1994.

[3] D. Black. On the rationale of group decision-making. *The Journal of Political Economy*, pages 23–34, 1948.

[4] E. H. Clarke. Multipart Pricing of Public Goods. *Public Choice*, 11:17–33, 1971.

[5] O. Dekel, F. Fischer, and A.D. Procaccia. Incentive compatible regression learning. In *Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 884–893. Society for Industrial and Applied Mathematics Philadelphia, PA, USA, 2008.

[6] A. Gibbard. Manipulation of voting schemes: A general result. *Econometrica: Journal of the Econometric Society*, pages 587–601, 1973.

[7] T. Groves. Incentives in Teams. *Econometrica*, 41:617–631, 1973.

[8] E. Miyagawa. Locating Libraries on a Street. *Social Choice and Welfare*, 18(3):527–541, 2001.

[9] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[10] A.D. Procaccia and M. Tennenholtz. Approximate mechanism design without money. In *Proceedings of the 10th ACM Conference on Electronic Commerce*, 2009.

[11] M.A. Satterthwaite. Strategy-proofness and Arrows conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

[12] J. Schummer and R. V. Vohra. *Mechanism design without money*, chapter 10. Cambridge University Press, 2007.

[13] Y. Sprumont. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. *Econometrica: Journal of the Econometric Society*, pages 509–519, 1991.

[14] W. Vickrey. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance*, 16:8–37, 1961.