Oblivious Online Contention Resolution Schemes

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Overview

- Online Contention Resolution Scheme (OCRS) is a framework that encompasses many stochastic optimization problems
- This work studies the possibility of OCRS when the underlying prior is unknown
    - A positive result in the simplest setting
    - Negative results in more general settings

Contention Resolution Scheme

- Setup: universe U, downward closed feasible sets $\mathcal{F} \subseteq 2^U$
- Input: $\mathbf{x} \in [0,1]^U$, the marginal dist. of a dist. over $\mathcal{F}$,

  An element in S is active

  sample set $S \subseteq U$ s.t. each element i is in S independently with probability x_i
- For $c > 0$, a c-balanced contention resolution scheme (CRS) outputs $T \subseteq S$, so that $\Pr[i \in T] \geq \frac{1}{c} \cdot x_i$ for each i
- Originally formulated by Chekuri, Vondrák & Zenklusen (2014) for submodular function maximization

Online Contention Resolution Scheme

- Setup: universe U, downward closed feasible sets $\mathcal{F} \subseteq 2^U$
- Input: a distribution $\mathbf{x}$ over $\mathcal{F}$, and a sampled set S such that each element i is in S independently with probability x_i, but the elements of U arrive one by one to reveal their memberships in S
- Feldman, Svensson & Zenklusen (2016): For $c > 0$, a c-selectable online contention resolution scheme (OCRS) outputs $T \subseteq S$, deciding for each $i \in S$ its membership in T as it arrives, so that $\Pr[i \in T] \geq \frac{1}{c} \cdot x_i$ for each i

Significance of (O)CRS

- The distribution **x** can be seen as a correlated distribution that results from

CRS is intimately connected with correlation gap

  - solving a continuous relaxation of an optimization problem

CRS was invented for rounding

  - or solving an ex ante relaxation of an optimization problem

OCRS generates approximately optimal online mechanism

closely related to prophet inequalities, sequential posted pricing…

Example: The Singleton Setting

- Let $\mathscr{F}$ be the singleton subsets of U, then $\mathbf{x}$ is a distribution on U.
  - A 2-selectable OCRS: if element i is active and we have not accepted anything, accept i so that it is accepted w.p. $1/2$.

Tight

  - This is possible because, with probability $1 - \sum_{j \text{ before } i} \frac{x_j}{2} \geq \frac{1}{2}$, we have not accepted anything when i arrives.
    - So if i is active, we accept it with probability $\frac{x_i}{2 - \sum_{j \text{ before } i} x_j}$
- When $\mathscr{F}$ is a matroid, there is a 2-selectable OCRS. [Singla & Lee 2018]

Oblivious OCRS

- An OCRS is oblivious if it has no knowledge of the distribution **x**
- Besides being mathematically attractive…
  - Dughmi 2020: matroid secretary is reducible to $O(1)$-selectable matroid OCRS with correlated inputs and limited knowledge on **x**
    - Dughmi already showed such OCRS is impossible; Dughmi 2022 even removed the part "limited knowledge on **x**"
  - For prophet inequalities in the singleton setting, with unknown distributions, a single sample from each distribution recovers the optimal competitive ratio [Rubinstein, Wang, Weinberg, 2020]

Our Main Results

- What's the best oblivious OCRS in the singleton setting?

For the singleton setting, we obtain an oblivious OCRS that is e-selectable for any $\mathbf{x}$, and show e is the best possible.

- Is there an $O(1)$-selectable oblivious OCRS for matroids?

There are transversal matroids and graphic matroids for which no oblivious CRS can be $O(1)$-balanced. Such CRS is not possible even with $O(1)$ many samples.

Optimal Oblivious OCRS for Singleton

- Intuition: how do we emulate the 2-selectable OCRS without knowing **x**?
  - When we see the first active element *i*, elements before *i* may be seen as samples from **x**
  - Since no element before *i* is active, we may estimate $\sum_{j \text{ before } i} x_j$ to be 0, and accept *i* with probability 1/2
  - If we did not accept *i*, then by the time we see the second active element *i*′, we may estimate $\sum_{j \text{ beofre } i'} x_j$ to be 1, and accept *i*′ with probability 1.

Optimal Oblivious OCRS for Singleton

- Algorithm: with probability 1/2, accept the first active element; otherwise, accept the second.

Theorem. This algorithm is e-selectable.

Proof. For element i, let $A(i)$ be the set of elements arriving before i.

When i is active, it is accepted with probability

$$\frac{1}{2}\left[\prod_{j\in A(i)}(1-x_j)+\sum_{j\in A(i)}x_j\prod_{k\in A(i)\setminus\{j\}}(1-x_k)\right], \text{ which minimizes to } \frac{1}{e} \text{ at } \mathbf{x}=(\frac{1}{n},\ldots,\frac{1}{n}).$$

Oblivious OCRS for Singleton: Lower Bound

- Algorithm: with probability 1/2, accept the first active element; otherwise, accept the second.

Theorem. No oblivious OCRS can be $(e - \epsilon)$-selectable for any $\epsilon > 0$.

Proof idea: I. An OCRS is counting based if for any active i,

$$\Pr[i \text{ accepted}] = f(\#\text{active elements before } i).$$

Not hard to show a counting based OCRS can't be $(e - \epsilon)$-selectable.

Oblivious OCRS for Singleton: Lower Bound

Theorem. No oblivious OCRS can be $(e - \epsilon)$-selectable for any $\epsilon > 0$.

Proof idea: I. A counting based OCRS can't be $(e - \epsilon)$-selectable.

II. For any OCRS, there is a subset S on which it is approximated by a counting based OCRS.

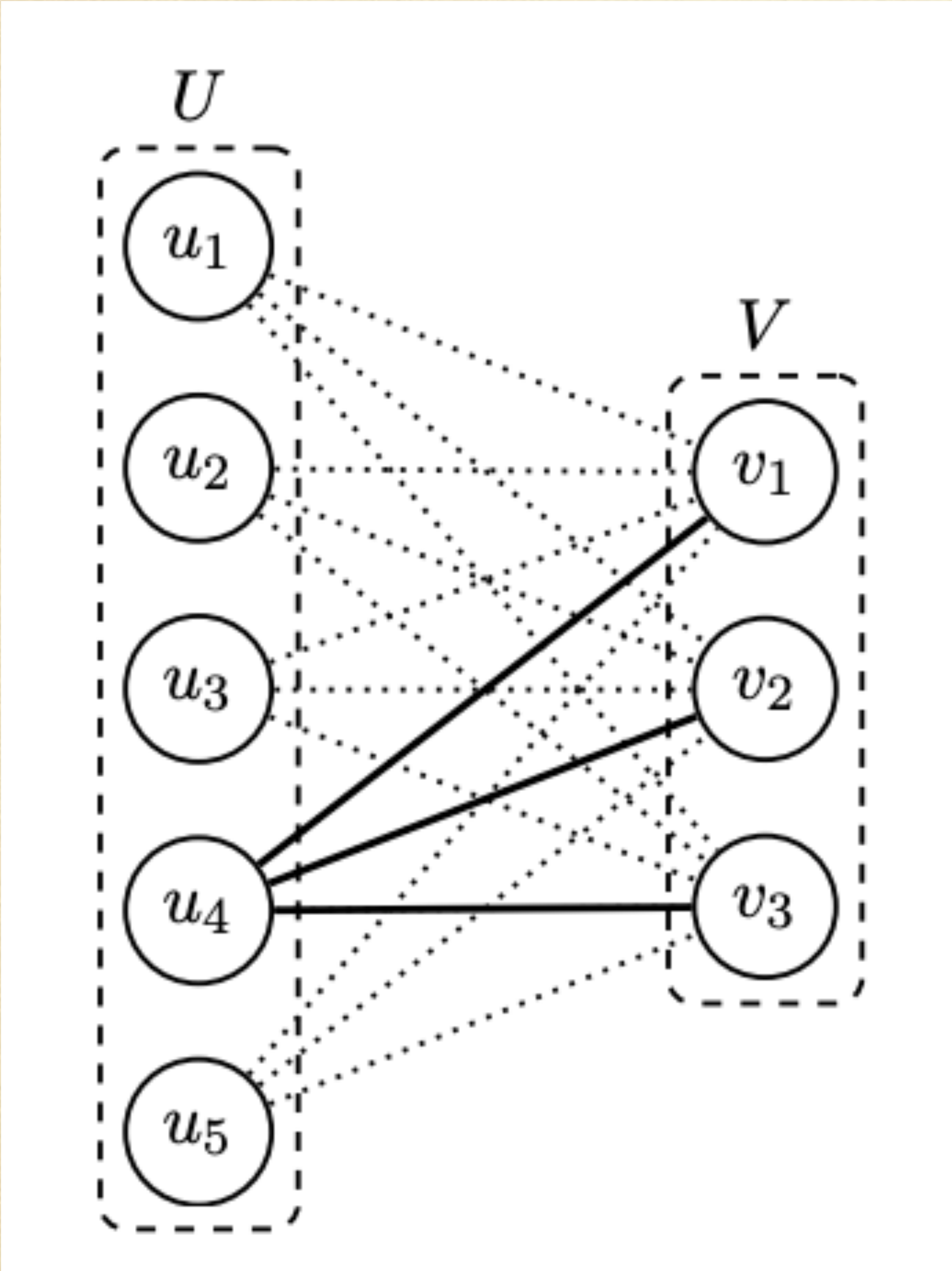
- Large enough n guarantees S of any desired size
- This is proved via a Hypergraph Ramsey theorem.
- This proof technique was used by Correa, Dütting, Fischer, Schewior 2019

Impossibility of Oblivious Matroid CRS

Thm. For any $c > 0$, there is no c-selectable oblivious CRS for graphic or transversal matroids.

Proof sketch (for graphic matroid). For a complete bipartite graph $K_{N,M}$, the weak distribution is one where each edge has weight $\frac{1}{M}$.

A strong distribution is hardwired with an event that is rare in the weak distribution.

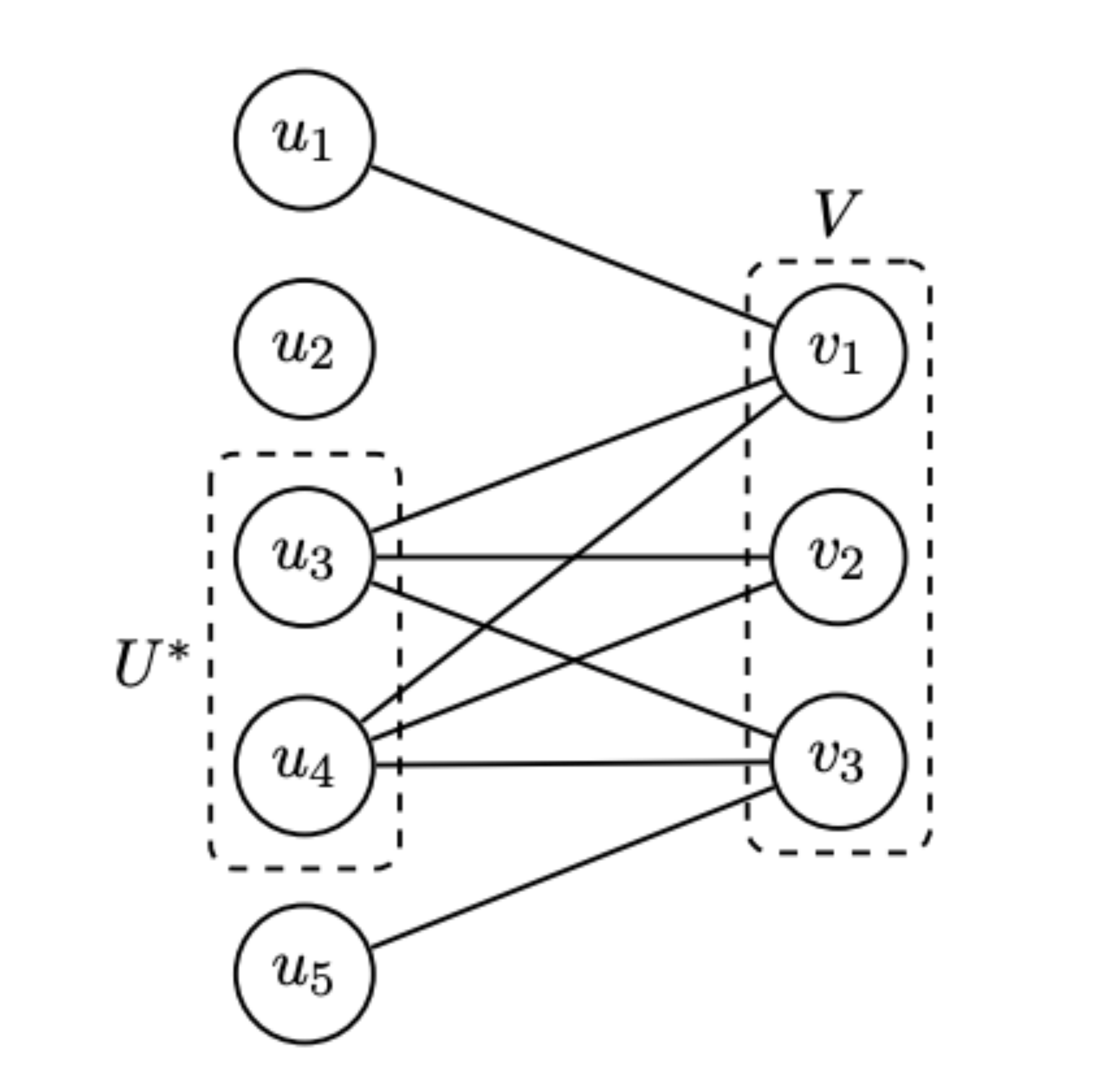


A strong distribution

The bold edges are active w.p. 1

Relationship:

Conditioning on that a sample from the weak distribution has all the edges in $\delta(u_i)$, the posterior distribution is precisely the strong distribution with $\delta(u_i)$ hardwired.

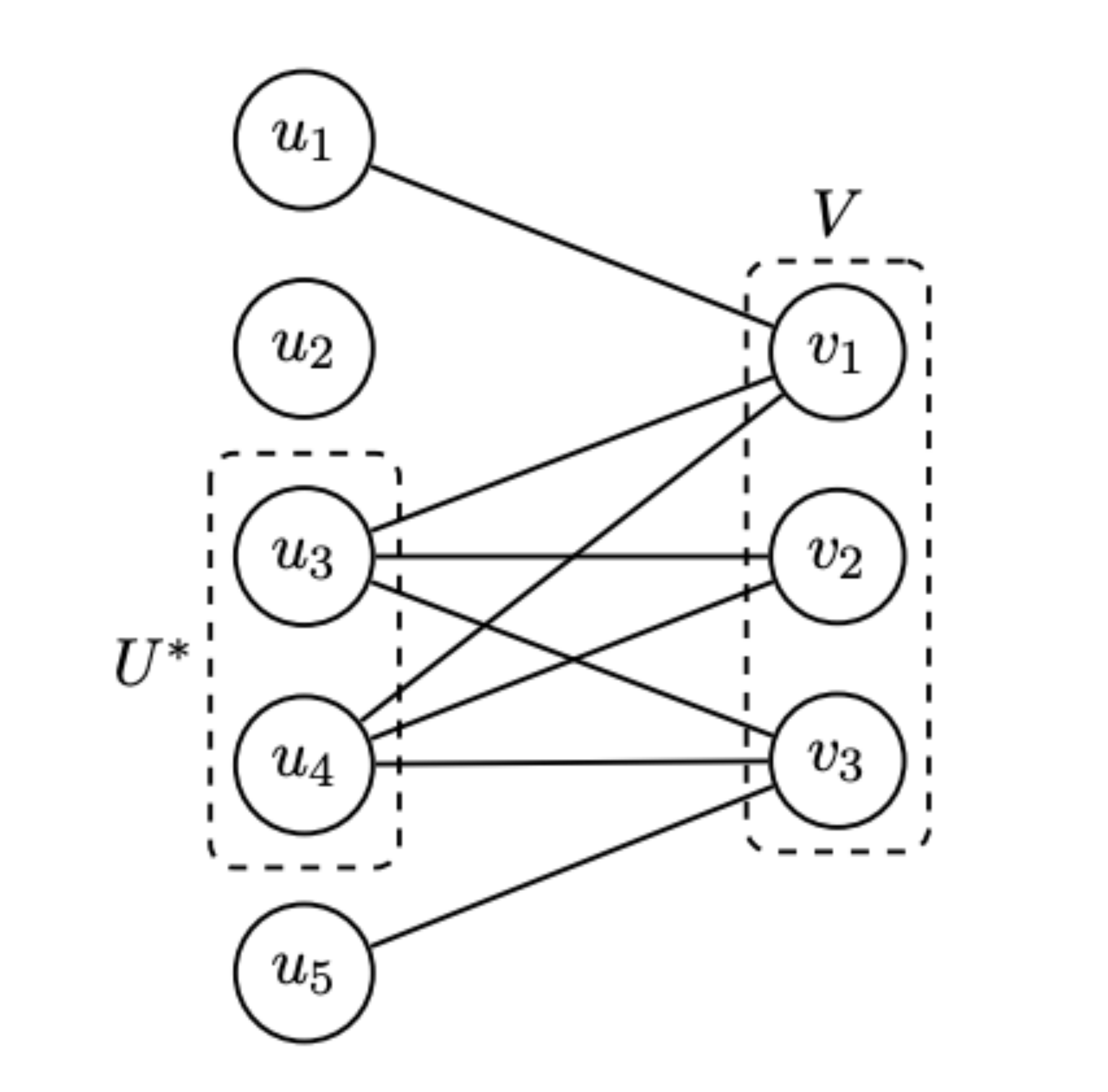


#edges accepted in $\delta(U^*)$:

$$\leq \text{rank}(\delta(U^*)) = |U^*| + M - 1$$

A sample from the weak distribution

U^*: left nodes with all incident edges active



A sample from the weak distribution

U^*: left nodes with all incident edges active

$\mathbb{E}$[#edges accepted in $\delta(U^*)$]:

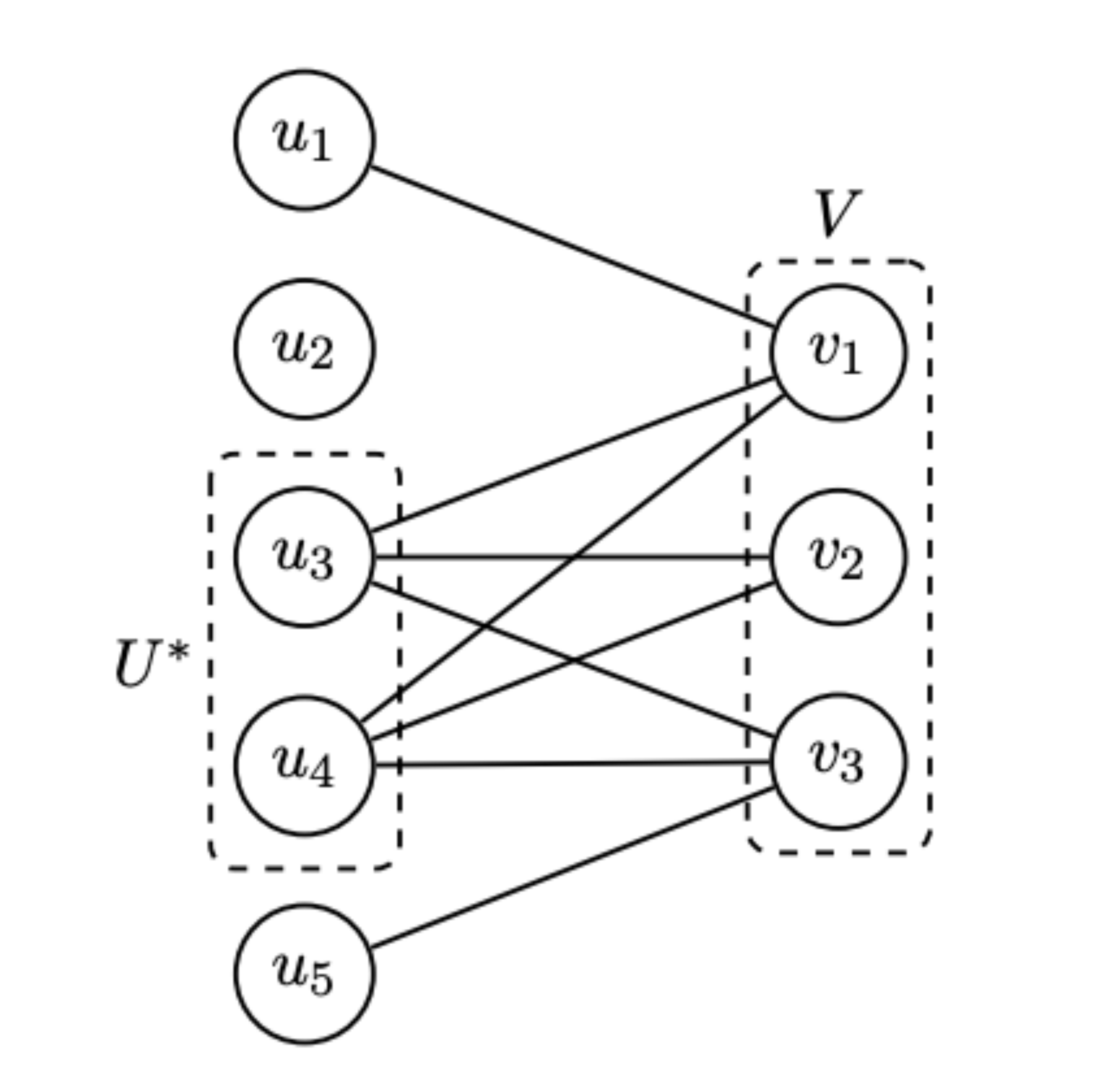
$$\leq \mathbb{E}[\text{rank}(\delta(U^*))] = \frac{N}{M^M} + M - 1$$

$$\geq NM \cdot \frac{1}{M^M} \cdot c$$

selectability on strong distribution

Total #edges

Pr[left endpoint in U^*]



A sample from the weak distribution

U^*: left nodes with all incident edges active

$\mathbb{E}[\#\text{edges accepted in } \delta(U^*)]$:

$$\leq \mathbb{E}[\text{rank}(\delta(U^*))] = \frac{N}{M^M} + M - 1$$

$$\geq NM \cdot \frac{1}{M^M} \cdot c$$

$$\Rightarrow c \leq \frac{1}{M} + \frac{M^{M-1}(M-1)}{N}$$

Make M, N arbitrarily large!

Impossibility of Oblivious Matroid CRS

Thm. For any $c > 0$, there is no c-selectable oblivious CRS for graphic or transversal matroids.

Proof sketch (for graphic matroid). For a complete bipartite graph $K_{N,M}$, the weak distribution is one where each edge has weight $\frac{1}{M}$.

A strong distribution is hardwired with an event that is rare in the weak distribution.

Remark: $O(1)$ samples cannot distinguish weak and strong distributions.

Conclusions

- Our results
  - An optimal oblivious OCRS for the singleton setting
  - $O(1)$-selectable oblivious OCRS for matroids is not possible
- Open questions
  - Sample complexity for $O(1)$-selectable matroid OCRS
  - Sample complexity for $O(1)$-competitive matroid prophet inequalities