

# Polynomial-Time Solution to the Hidden Subgroup Problem for a Class of non-abelian Groups

Martin Rötteler<sup>1</sup>  
roettele@ira.uka.de

Thomas Beth  
EISS\_Office@ira.uka.de

Institut für Algorithmen und Kognitive Systeme  
Universität Karlsruhe, Germany

## Abstract

We present a family of non-abelian groups for which the hidden subgroup problem can be solved efficiently on a quantum computer.

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<sup>1</sup>supported by DFG grant GRK 209/3-98

# 1 Introduction

The hidden subgroup problem has found recent interest in the theory of quantum computing. This is due to the fact that the power of quantum computation compared to classical computation becomes apparent in this problem.

The first occurrence of a hidden subgroup problem for an abelian group appeared has been implicitly in Simon's work [14]. In fact he solved the hidden subgroup problem for the group  $Z_2^n$  under the promise that the hidden subgroup is of order 2. The group theoretical interpretation of this algorithm has been formulated by several authors (see [3], [9], [10]). In the paper [3] it is shown that subgroups of arbitrary order can be found and furthermore that this can be done by an exact quantum polynomial time algorithm. Thus there is an exponential speed-up of this quantum algorithm over any classical algorithm, even probabilistic ones.

Recently the question has been raised as to whether the hidden subgroup problem could also be solved for non-abelian groups. In the paper [4] the problem is addressed for the dihedral groups  $D_N$ . The authors have found an interesting way to circumvent the application of the Fourier transform for the dihedral groups and instead use the Fourier transform for  $Z_N \times Z_2$  and still learn something from the probability distribution about the existence or non-existence of certain elements. However the classical post-processing requires an optimization problem which makes the overall algorithm exponential in the number of classical steps, but is polynomial in the number of evaluations of the quantum black-box circuit representing the given function.

In this paper we will present a family  $W_n$  of non-abelian groups for which the hidden subgroup problem can be solved by a number of steps polynomial in the number of qubits. The groups in this family are certain semi-direct products (namely wreath products) and have some desirable properties (they are, e. g., of bounded exponent). Moreover, the fact that a Fourier transform for  $W_n$  can be performed efficiently by a quantum computer is important to solve the hidden subgroup problem for these groups.

## 2 The Hidden Subgroup Problem

We adopt the definition of the hidden subgroup problem given in [4]. The history of the hidden subgroup problem parallels the history of quantum computing since the algorithms of Simon [14] and Shor [13] can be formulated in the language of hidden subgroups (see e. g. [9] for this reduction) for certain abelian groups. In the paper [3] an exact quantum algorithm (running in polynomial time in the number of evaluations of the given black box function and the classical post-processing) is given for the hidden subgroup problem in the abelian case.

**Definition 2.1 (The hidden subgroup problem)**

Let  $G$  be a finite group and  $f : G \rightarrow R$  a mapping from  $G$  to an arbitrary domain  $R$  fulfilling the following conditions:

- a) The function  $f$  is given as a quantum circuit, i. e.,  $f$  can be evaluated in superpositions.
- b) There exists a subgroup  $U \subseteq G$  such that  $f$  takes a constant value on each of the cosets  $gU$  for  $g \in G$ .
- c) Furthermore  $f$  takes different values on different cosets.

The problem is to find generators for  $U$ .

### 3 Wreath Products

In this section we recall the definition of wreath products in general (see also [6] and [8]) and define the family of groups for which we will solve the hidden subgroup problem.

**Definition 3.1** Let  $G$  be a group and  $H \subseteq S_n$  be a subgroup of the symmetric group on  $n$  letters. The wreath product  $G \wr H$  of  $G$  with  $H$  is the set

$$\{(\varphi, h) : h \in H, \varphi : [1, \dots, n] \rightarrow G\}$$

equipped with the multiplication

$$(\varphi_1, h_1) \cdot (\varphi_2, h_2) := (\psi, h_1 h_2),$$

where  $\psi$  is the mapping which sends  $i \mapsto \varphi_1(i^{h_2})\varphi_2(i)$  for  $i \in [1, \dots, n]$ .

The wreath product is isomorphic to a semidirect product of the so-called *base group*  $N := G \times \dots \times G$  which is the  $n$  fold direct product of (independent) copies of  $G$  with  $H$ , in symbols  $G \wr H = N \rtimes H$ , where  $H$  operates via permutation of the direct factors of  $N$ . So we can think of the elements to be  $n$ -tuples of elements from  $G$  together with a permutation  $\tau$  and multiplication is done component-wise after a suitable permutation of the first  $n$  factors

$$(g_1, \dots, g_n; \tau) \cdot (g'_1, \dots, g'_n; \tau') = (g_{\tau'(1)}g'_1, \dots, g_{\tau'(n)}g'_n; \tau\tau').$$

## 4 The Wreath Products $Z_2^n \wr Z_2$

In the following we show some elementary properties of  $W_n := Z_2^n \wr Z_2$ . The groups  $W_n$  have exponent 4 and base-group  $N := Z_2^n \times Z_2^n$ . Elements of  $W_n$  are denoted by  $(x, y; a)$  where  $x, y \in Z_2^n$  and  $a \in Z_2$ . For a subgroup  $U$  and an element  $g \in W_n$  as usual we define  $U^g := \{g^{-1}ug : u \in U\}$ . We think of the elements encoded in such a way that  $x$  and  $y$  are encoded in the lower significant bits and  $a$  is the most significant bit. We later need the important

**Lemma 4.1** *Let  $U$  be a subgroup of  $W_n$  and  $t = (0, 0; 1)$ . Then*

$$U = (U \cap N) \cdot (U \cap U^t). \quad (1)$$

**Proof:** "⊇" is clear since  $U \cap N$  and  $U \cap U^t$  are subgroups of  $U$ .

"⊆": Let  $u \in U$  be a given element. Since  $u \in N$  implies that  $u$  is contained in the left factor, we can assume that  $u \notin N$ , i. e.,  $u = (x, y; 1)$  for certain  $x, y \in Z_2^n$ . We compute

$$u^2 = (x \oplus y, x \oplus y; 0), \quad u^3 = (y, x; 1), \quad u^4 = (0, 0; 0).$$

Thus the effect of conjugating  $u$  with  $t$  is  $u^t = (y, x; 1)$  from which we can deduce  $u \in U^t$  since  $u = (u^3)^t \in U^t$ .  $\square$

**Remark 4.2** a) The preceding lemma shows that each subgroup  $U$  of  $W_n$  factorizes in a canonical way into the product of two subgroups. Therefore it is sufficient to find generators for  $U \cap N$  and  $U \cap U^t$  to obtain a set of generators for  $U$ .

b) The action on an element  $n = (x', y'; 0) \in N$  of an arbitrary transversal element  $\tau = (x, y; 1)$  for which

$$W_n \begin{matrix} \{1, \tau\} \\ \triangleright \\ N \triangleright E \end{matrix}$$

holds, is given by

$$\tau^{-1}n\tau = (y, x; 1)(x', y'; 0)(x, y; 1) = (y, x; 1)(x \oplus y', y \oplus x'; 1) = (y', x'; 0),$$

i. e., the components of  $n$  are swapped.

c) From the isomorphism theorem for  $U \subseteq W_n$  follows

$$NU/N \cong U/U \cap N.$$

Thus the index of  $U \cap N$  in  $U$  can be 1 or 2, since  $N$  is a maximal normal subgroup of  $W_n$ . So the subgroup  $U \cap U^t$  indicates whether the index is 1 or 2 and the case of index 2 occurs iff there exists an element of the form  $(x, y; 1)$  in  $U$ .

- d) If the index  $[U : U \cap N] = 2$  then  $(U \cap N)^t = U \cap N$ . This is readily seen from b) by observing the fact that there must exist an element of the form  $(a, b; 1) \in U$ .
- e) We call subgroups fulfilling the property  $U = U^t$  **balanced**. Later on the balanced subgroups of  $U$  will play an important rôle since they will appear naturally in the process of sampling.

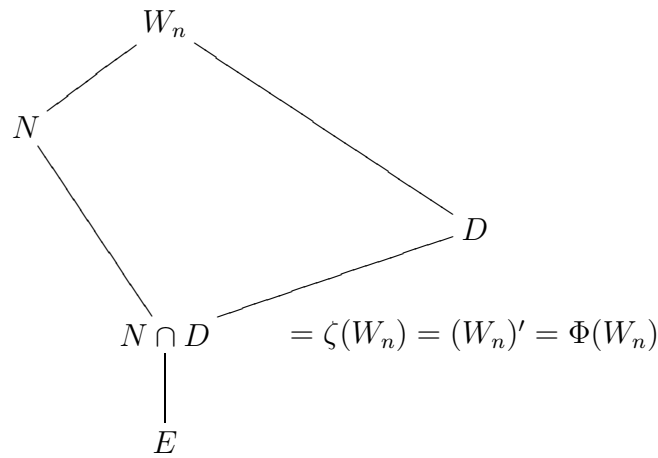


Figure 1: The wreath product  $W_n$  factors over  $N$  and  $D$

#### 4.1 Finding Involutions in $W_n$

In this section we present a straightforward method to find hidden subgroups of order 2 in  $W_n$ ; i. e., if it is promised that  $U$  has order 2, then the generator of  $U$  can be found without invoking non-abelian Fourier transforms.

Consider the restriction of  $f$  to the base group  $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ : We can obtain an equal distributed superposition over the base group  $N$  by application of the  $2 \times 2$  Hadamard matrix  $H$  on all qubits except for the most significant one.

Then we use the quantum algorithm for the hidden subgroup problem for  $\mathbb{Z}_2^{2n}$  using only the first  $2n$  bits. This gives generators for the group  $U \cap N$ , so that the number of evaluations of  $f$  is linear in  $n$  (also the classical post-processing needs only a number of steps which is linear in  $n$ ).

Note that each element in  $N$  has order 2 but there may be more involutions in  $W_n$ . More precisely: Each involution is contained in  $N$  or  $D$  where  $D$  is defined by

$$D = \{(x, x; a), \text{ where } x \in \mathbb{Z}_2^n \text{ and } a \in \mathbb{Z}_2\}$$

This is because of the observation that for an element  $(x, y; a) \in W_n$

$$(x, y; a)^2 = (x \oplus y, x \oplus y, 0) \stackrel{!}{=} (0, 0; 0) \Rightarrow x = y$$

holds. We also denote  $D$  by  $(\mathbb{Z}_2^n \parallel \mathbb{Z}_2^n) \times \mathbb{Z}_2$  since the two factors are diagonal.

Figure 1 shows the situation involving  $W_n$ ,  $N$  and  $D$ . Interestingly, the intersection of  $N$  and  $D$  is the center  $\zeta(W_n)$  of  $W_n$  which coincides with the commutator  $W_n'$  and the Frattini subgroup  $\Phi(W_n)$  of  $W_n$  but these facts will not be used in the sequel.

Since  $D$  is abelian we can solve the hidden subgroup problem for  $D$  by the usual abelian hidden subgroup algorithm. We do this by performing Hadamard transforms on the qubits representing  $y$  and  $a$  followed by controlled NOTs between qubits  $x_i$  and  $y_i$  for  $i = 1, \dots, n$  (see figure (2)). This is followed by the hidden subgroup algorithm for  $\mathbb{Z}_2^{n+1}$  applied to the bits representing  $y$  and  $a$ .

## 4.2 The Pairing on $W_n$

In view of the Fourier transform for  $W_n$  to come we define (mimicking the abelian case) a pairing  $\mu$  on  $W_n$  which in turn allows the definition of "duals" needed to treat this case of non-abelian groups.

**Definition 4.3** *We denote by  $\mu : W_n \times W_n \rightarrow \mathbb{Z}_2$  the pairing*

$$\mu((x, y; a), (x', y'; a')) := \begin{cases} \sum x_i x'_i + \sum y_i y'_i & : a = a' = 0 \\ \sum x_i y'_i + \sum x'_i y_i & : a \oplus a' = 1 \\ \sum x_i x'_i + \sum y_i y'_i + 1 & : a = a' = 1 \end{cases}$$

Like in the abelian case we denote suggestively the set of perpendicular elements for a given  $U \subseteq W_n$  by

$$U^\perp := \{g \in W_n : \forall h \in U : \mu(g, h) = 0.\}$$

However, in general  $U^\perp$  will not be a group any more. We will also make use of a bijective mapping (which is of course not a homomorphism)  $\varphi : W_n \rightarrow \mathbb{F}_2^{2n+1}$  which sends

$$\varphi : \begin{array}{l} (x, y; 0) \mapsto (x, y, 0) \\ (x, y; 1) \mapsto (y, x, 1) \end{array}, \quad \text{where } x, y \in \mathbb{Z}_2^n.$$

Using  $\varphi$  we can compute  $\mu$  using the identity

$$\forall g, h \in W_n : \mu(x, y) = \langle \varphi(g), \varphi(h) \rangle_{\mathbb{F}_2^{2n+1}}$$

which holds due to the construction of  $\mu$  and  $\varphi$ .

Here and in the following we let  $t$  denote the element  $(0, 0; 1) \in W_n$ .

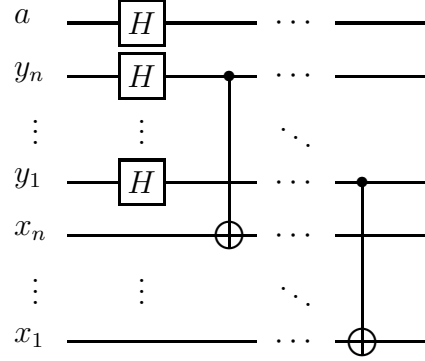


Figure 2: Finding involutions in  $D$

**Lemma 4.4** For a subgroup  $U$  of  $W_n$  the following holds:

$$y \notin U \Rightarrow \sum_{x \in U} \mu(x, y) = 0.$$

**Proof:** Write  $U = (U \cap N) \cdot (U \cap U^t)$ . Two cases can occur:

1.)  $U = U \cap N$  (i. e.  $U$  is abelian). Then

$$\varphi(U) = \{(x_i, y_i, 0) \in \mathbb{F}_2^{2n+1} : (x_i, y_i, 0) \in U\}.$$

This is a linear subspace of  $\mathbb{F}_2^{2n+1}$  and since  $y \notin U$  there exists  $v \in \varphi(U)$  such that  $\langle \varphi(y), v \rangle \neq 0$ . Invoking an  $\mathbb{F}_2$ -vector space argument we conclude  $\sum_{u \in \varphi(U)} \langle \varphi(y), u \rangle = 0$ .

2.)  $U = (U \cap N) \dot{\cup} (U \cap N) \cdot t_0$  with  $t_0 = (a, b, 1)$ . Due to the preceding remark we have  $(U \cap N)^t = (U \cap N)$ , i. e.,  $\{(x_i, y_i, 0) \in U \cap N\} = \{(y_i, x_i, 0)\}$  and mapping via  $\varphi$  yields the decomposition of  $\varphi(U)$  into a vector space  $V$  and an affine space

$$\varphi(U) = \underbrace{\{(x_i, y_i, 0)\}}_{=:V} \dot{\cup} (b, a, 1) \oplus \underbrace{\{(y_i, x_i, 0)\}}_{=:V},$$

since  $V$  is a balanced.

If there exists, an element  $x_0 \in V$  with  $y \notin x_0$  then  $\sum_{x \in V} \langle \varphi(y), x \rangle = 0$  and also  $\sum_{x \in V} (\langle \varphi(y), (b, a, 1) \rangle + \langle \varphi(y), x \rangle) = 0$ .

If no such element exists  $y$  is perpendicular on  $V$  and therefore necessarily  $\langle \varphi(y), (b, a, 1) \rangle = 1$ . This means that  $\sum_{x \in V} (\langle \varphi(y), (b, a, 1) \rangle + \langle \varphi(y), x \rangle) = \sum_{x \in V} \langle \varphi(y), (b, a, 1) \rangle = 0$ .  $\square$

We state another useful property of  $\mu$  which will be needed later on.

**Lemma 4.5** Let  $U$  be a subgroup of  $W_n$ . If there exists an element of the form  $(x, y, 1) \in U^\perp$  then exactly half of the elements of  $U^\perp$  are in  $N$ .

**Proof:** This follows from the fact that for  $g \in W_n$ ,  $h = (x, y; 1)$  and  $u = (x', y'; 0)$  we have:

$$\mu(g, u \cdot h) = \mu(g, u) \oplus \mu(g, h),$$

which follows from an easy computation.  $\square$

### 4.3 The Lattice of Balanced Subgroups

We have introduced the pairing  $\mu$  on  $W_n$  with respect to which we can define orthogonal complements. However, as stated, for a given  $U$  the orthogonal complement  $U^\perp$  need not again be a group.

For example in case of  $W_1$  we have

$$U = \{(0, 0; 1), (0, 1; 0)\}, \quad U^\perp = \{(0, 0; 0), (0, 0; 1), (0, 1; 1), (1, 0; 0)\}$$

and  $(0, 1; 1)^2 = (1, 1; 0) \notin U^\perp$ . But we have the following

**Theorem 4.6** *Let  $U \subseteq W_n$  be a subgroup and  $t = (0, 0; 1)$ . Then*

$$U = U^t \Leftrightarrow U^\perp \text{ is a subgroup of } W_n.$$

**Proof:** "⇒": By looking at the linear equations defining  $U^\perp$  when we employ the bijection  $\varphi$ , we firstly observe that  $U^\perp$  is again balanced:

$$\begin{bmatrix} \vdots \\ x_i, y_i, a_i \\ \vdots \\ y_i, x_i, a_i \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_{2^{2n+1}} \end{bmatrix} = 0. \quad (2)$$

If  $(x', y', a')$  is a solution of (2) then also  $(y', x', a')$  is a solution, since with each row  $(x_i, y_i, a_i)$  we have also the row  $(y_i, x_i, a_i)$  appearing.

Now let  $g = (x_1, y_1, a_1)$  and  $h = (x_2, y_2, a_2)$  be given elements from  $U^\perp$ .

$$\varphi(g \cdot h) = \begin{cases} (x_1, y_1, a_1) \oplus (x_2, y_2, a_2), & \text{if } a_2 = 0 \\ (y_1, x_1, a_1) \oplus (x_2, y_2, a_2), & \text{if } a_2 = 1. \end{cases}$$

Since  $g \in U^\perp$  also  $(y_1, x_1, a_1) \in U^\perp$ , so for all  $u \in U$  the following holds:

$$\begin{aligned} \langle \varphi(g \cdot h), u \rangle &= \begin{cases} \langle (x_1, y_1, a_1), u \rangle + \langle (x_2, y_2, a_2), u \rangle, & \text{if } a_2 = 0 \\ \langle (y_1, x_1, a_1), u \rangle + \langle (x_2, y_2, a_2), u \rangle, & \text{if } a_2 = 1 \end{cases} \\ &= 0 \end{aligned}$$



Therefore  $g \cdot h \in U^\perp$ . Closedness under taking inverses follows from the fact that elements  $u \in W_n$  are either involutions or  $u^3 = u^{-1}$ .

" $\Leftarrow$ ": It is sufficient to show that  $U^\perp$  is balanced since  $(U^\perp)^\perp = U$ . Without loss of generality we can assume that  $U \subseteq N$ , since otherwise there exists  $(a, b; 1) \in U$  from which we can deduce  $u^{(a,b;1)} = u^t$  for all  $u \in U$  and we will be done.

So we have to show that there exists  $(a, b; 1) \in U^\perp$  (then  $U^\perp$  will by the same argument be balanced and correspondingly  $U$ , too). Looking at the equations

$$\begin{bmatrix} \vdots \\ x_i, y_i, a_i \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_{2^{2n+1}} \end{bmatrix} = 0$$

we see that such an element must exist since if  $(z_1, \dots, z_{2^{2n}}, 0)$  is a solution, then  $(z_1, \dots, z_{2^{2n}}, 1)$  will also be a solution.  $\square$

The following corollary summarizes some further properties of the pairing  $\mu$ .

**Corollary 4.7** a) For all subgroups  $U \subseteq W_n$

$$(U^t)^\perp = (U^\perp)^t.$$

b) Complements of intersections:

$$(U \cap U^t)^\perp = \langle U^\perp, (U^t)^\perp \rangle$$

c) The balanced subgroups of  $W_n$  correspond one-to-one to the balanced subspaces of  $\mathbb{F}_2^{2^{2n+1}}$ .

d) There is an inclusion-reversing anti-isomorphism  $\perp$  on the lattice of balanced subgroups of  $W_n$  which is a Galois correspondence.

**Proof:** a) Follows from the fact that  $\mu(x, x') = \mu(x^t, x'^t)$  for all  $x, x' \in W_n$ . b) follows from linear algebra over  $\mathbb{F}_2$ , c) is just a reformulation of lemma 4.6 and d) is obvious.  $\square$

## 5 Fourier Transforms for Wreath Products

In this section we show how to compute a Fourier transform for the groups  $W_n$  effectively on a quantum computer. We want to do this in brief since the general recursive method to obtain fast Fourier transforms on a quantum computer described in [12] can be applied directly in case of wreath products  $A \wr \mathbb{Z}_2$  where  $A$  is an arbitrary abelian 2-group (for efficient quantum transforms see also [5]).

The recursion of the algorithm follows the chain

$$A \wr Z_2 \triangleright A \times A \triangleright E,$$

where the second composition factor is the base group. We first want to determine the irreducible representations of  $G := A \wr Z_2$ . Let  $G^*$  be the base group of  $G$ , i.e.  $G^* = A \times A$ .  $G^*$  is a normal subgroup of  $G$  of index 2. Denoting by  $\mathcal{A} = \{\chi_1, \dots, \chi_k\}$  the set of irreducible representations of  $A$  recall that the irreducible representations of  $G^*$  are given by the set  $\{\chi_i \otimes \chi_j : i, j = 1, \dots, k\}$  of pairwise tensor products (see, e.g., [7] section 5.6).

Since  $G^* \triangleleft G$  the group  $G$  operates on the representations of  $G^*$  via inner conjugation. Because  $G$  is a semidirect product of  $G^*$  with  $Z_2$  we can write each element  $g \in G$  as  $g = (a_1, a_2; \tau)$  with  $a_1, a_2 \in A$  and we conclude

$$(\chi_1 \otimes \chi_2)^g = (\chi_1^{a_1} \otimes \chi_2^{a_2})^\tau = (\chi_1 \otimes \chi_2)^\tau,$$

i.e., only the factor group  $G/G^* = Z_2$  operates via permutation of the tensor factors. The operation of  $\tau$  is to map  $\chi_1 \otimes \chi_2 \mapsto \chi_2 \otimes \chi_1$ .

Therefore it is easy to determine the inertia groups (see [6], [2] for definitions)  $T_\rho$  of a representation  $\rho$  of  $G^*$ . We have to consider two cases:

- a)  $\rho = \chi_i \otimes \chi_i$ . Then  $T_\rho = G$  since permutation of the factors leaves  $\rho$  invariant.
- b)  $\rho = \chi_i \otimes \chi_j, i \neq j$ . Here we have  $T_\rho = G^*$ .

The irreducible representations of  $G^*$  fulfilling a) extend to representations of  $G$  whereas the induction of a representation fulfilling b) is irreducible. In this case the restriction of the induced representation to  $G^*$  is by Clifford theory equal to the direct sum  $\chi_1 \otimes \chi_2 \oplus \chi_2 \otimes \chi_1$ .

Applying the design principles for Fourier transforms given in [12] we obtain the circuits for  $\text{DFT}_{W_n}$  in a straightforward way. In doing so it is necessary to study the extension/induction behaviour of representations of  $G^*$  since the recursive formula

$$\text{DFT}_{G^*} \cdot \bigoplus_{t \in T} \Phi(t) \cdot \text{DFT}_{Z_2}$$

provides a Fourier transform for  $G$ . Here  $\Phi(t)$  denotes the extension (as a whole) of the regular representation of  $G^*$  to a representation of  $G$  (see [11], [2], [12]). In case of  $W_n$  the transform  $\text{DFT}_{G^*}$  is the Fourier transform for  $Z_2^{2n}$  and therefore a tensor product of  $2n$  Hadamard matrices.

The circuits for the case of  $W_n$  are shown in figure 3. Quantum circuits in general are built from certain gate primitives (see [1]) and it is clear that the complexity cost for this circuit is linear in the number of qubits, since the conditional gate representing the evaluation at the transversal  $\bigoplus_{t \in T} \Phi(t)$  can be realized with  $3n$  Toffoli gates.

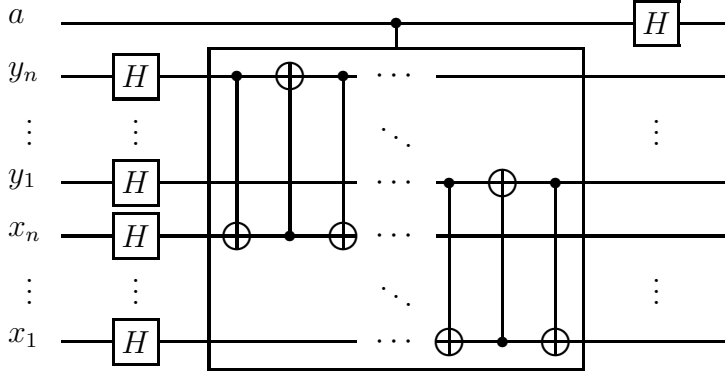


Figure 3: The Fourier transform for  $Z_2^n \wr Z_2$

Finally we give a slight modification of this circuit by performing the same matrix  $\bigoplus_{t \in T} \Phi(t)$  (which in this case is a permutation matrix) at the end yielding

$$\text{DFT}_{W_n} := \text{DFT}_{G^*} \cdot \bigoplus_{t \in T} \Phi(t) \cdot \text{DFT}_{Z_2} \cdot \bigoplus_{t \in T} \Phi(t). \quad (3)$$

This again decomposes the regular representation of  $G$  into irreducibles and has the advantage to allow a reinterpretation of the pairing  $\mu$  given in section 4.2 for the groups  $W_n$ :

Multiplying the matrices in (3) yields (the permutation matrix  $\Pi$  exchanges the qubits  $x_i$  and  $y_i$  for  $i = 1, \dots, n$ )

$$\text{DFT}_{W_n} = \left( \begin{array}{c|c} H^{\otimes 2n} & H^{\otimes 2n} \cdot \Pi \\ \hline H^{\otimes 2n} \cdot \Pi & -H^{\otimes 2n} \end{array} \right),$$

and therefore the matrix entry  $\text{DFT}_{[g,h]}$  equals  $(-1)^{\mu(g,h)}$  for all  $g, h \in W_n$ , where we use the already mentioned enumeration of the group elements and the pairing  $\mu$  defined in 4.3.

## 6 Sampling the Fourier Coefficients

In this section we address the problem to gain enough information from the Fourier coefficients under  $\text{DFT}_{W_n}$  to find generators for the  $U \cap U^t$  part from factorization (1). The idea is to find generators for the balanced group  $\langle U^\perp, (U^\perp)^t \rangle$  from which we get generators for  $U \cap U^t$  by taking orthogonal complements.

First we want to describe the elements in  $U^\perp$ :

**Remark 6.1** *Let  $U \subseteq W_n$ . Then one of the following cases holds:*

- a)  $U^\perp$  consists exclusively of elements of the form  $(x, y; 0)$ , i. e.  $U^\perp = U^\perp \cap N$ . Since  $U^\perp \cap N$  is a subgroup of  $W_n$  it follows that  $U^\perp$  is a group, thus sampling

from an equal distribution over  $U^\perp$  will give generators after a few steps (for an exact analysis see below).

- b) There exists an element  $t_0 = (x, y; 1)$  in  $U^\perp$ . We can conclude that exactly half of the elements are in  $U \cap N$  and the other half is of the form  $t_0 \cdot U \cap N$  (see lemma 4.5).

The following theorem shows that the Fourier transform for the groups  $W_n$  have properties very similar to the abelian case:

**Theorem 6.2** Let  $\text{DFT}_{W_n} = \sum_{x,y \in W_n} \mu(x, y) |y\rangle \langle x|$  be the Fourier matrix. Then for each subgroup  $U \subseteq W_n$  we have:

$$\text{DFT}_{W_n} \frac{1}{|U|} \sum_{x \in U} |x\rangle = \frac{1}{|U^\perp|} \sum_{y \in U^\perp} |y\rangle.$$

**Proof:** Since

$$\begin{aligned} \text{DFT}_{W_n} \frac{1}{|U|} \sum_{x \in U} |x\rangle &= \left( \sum_{x,y \in W_n} \mu(x, y) |y\rangle \langle x| \right) \sum_{x \in U} |x\rangle \\ &= \sum_{y \in W_n} \sum_{x \in U} \mu(x, y) |y\rangle, \end{aligned}$$

it suffices to show  $\sum_{x \in U} \mu(x, y) = 0$  for  $y \notin U^\perp$ , but this statement is lemma 4.6. The other case  $\sum_{x \in U} \mu(x, y) = |U|$  for  $y \in U^\perp$  is obvious.  $\square$

Next we show that sampling yields also information about  $U$  in case we have drawn a coset  $g_0 U$  instead of  $U$ . In case  $g_0 \in N$  we indeed sample from  $U$ , since  $N$  acts diagonally in the Fourier basis with phase factors  $\pm 1$ , i. e.

$$\text{DFT}_{W_n} \frac{1}{|U|} \sum_{x \in g_0 U} |x\rangle = \frac{1}{|U^\perp|} \sum_{y \in U^\perp} \varphi_{g_0, y} |y\rangle$$

with certain phase factors  $\varphi_{g_0, y}$  which depend on  $g_0$  and  $y$  but are always from  $\{\pm 1\}$ . Since making measurements involves taking the squares of the amplitudes we get an equal distribution over  $U^\perp$ .

The other case  $g_0 \in W_n \setminus N$  leads to an equal distribution over  $(U^t)^\perp$ , since an element  $g_0 = nt, n \in N$  operates up to phase factors like  $t$  in the Fourier basis and  $t$  swaps  $(x, y; a)$  and  $(y, x; a)$  when considered as basis vectors of the Fourier basis.

## 6.1 Analysis of Sampling

By the preceding observations we are able to take samples equally distributed from the sets  $U^\perp$  and  $(U^t)^\perp$  according to whether  $g_0 \in N$  or  $g_0 \in W_n \setminus N$ . Both cases occur with probability  $1/2$  since  $[W_n : N] = 2$ . We now have to show that after a few samples we have found generators for the group  $\langle U^\perp, (U^t)^\perp \rangle$  generated by  $U^\perp$  and  $(U^t)^\perp$ .

We denote the set of sampled elements after the  $i$ -th measurement by  $\mathcal{E}_i$  and have to give a bound on the probability that  $\mathcal{E}_i$  generates this group.

**Lemma 6.3** *For the probability  $P$  of finding a set of generators after  $i$  samples we have the following estimation*

$$P(\langle \mathcal{E}_i \rangle = \langle U^\perp, (U^t)^\perp \rangle) \geq 1 - 2^{-i/4}.$$

**Proof:** We already know that  $U^\perp \cap N$  and  $(U^t)^\perp \cap N$  are groups. Also we know from remark 6.1 that if there is one element in  $U^\perp$  which is not in  $N$  then exactly half of the elements in  $U^\perp$  must be in  $N$  and the other half in  $W_n \setminus N$ . The same argument holds for  $(U^t)^\perp$ . Thus in the worst case we are facing the situation, that with each sample we fall into one of the boxes

$$\boxed{U^\perp \cap N} \quad \boxed{(U^\perp \cap N) \cdot t_0} \quad \boxed{(U^t)^\perp \cap N} \quad \boxed{((U^t)^\perp \cap N) \cdot t'_0}$$

(with certain elements  $t_0$  and  $t'_0$  from  $W_n \setminus N$ ). The probability not to have generated  $U^\perp \cap N$  and  $(U^t)^\perp \cap N$  after  $i$  steps is smaller than  $2^{-i/4}$ . From the other two sets only one element is necessary to discriminate between the index 1 and index 2 case and so the statement follows.  $\square$

One remark is in order, since it is necessary to have a criterion when to stop sampling: Arguing like in [3], suppose the group generated by  $\mathcal{E}_i$  is too small, i. e., after taking duals we are dealing with  $U' \supset U$ . Then one of the generators found must necessarily evaluate to a different value than the neutral element of  $U$  does. This is due to the promise about  $f$  and can be checked in polynomial time by comparing the values on all generators found.

## 7 The Quantum Algorithm

Using the results of the preceding sections we can now formulate a quantum algorithm which solves the hidden subgroup problem for the non-abelian groups  $W_n$ . It uses  $O(n)$  evaluations of the black box quantum circuit  $f$  and the classical post-computation, which is essentially linear algebra over  $\mathbb{F}_2$ , also takes a number of operations which is polynomial in  $n$ .

**Algorithm 7.1** 1. Prepare the ground state

$$|\varphi_1\rangle = |0 \dots 0\rangle \otimes |0 \dots 0\rangle$$

*in both registers.*

2. Achieve equal amplitude distribution in the first register, for instance by an application of a Hadamard transform to each qubit:

$$|\varphi_2\rangle = \sum_{x \in W_n} |x\rangle \otimes |0 \dots 0\rangle.$$

(Normalization factors omitted.)

3. Calculate  $f$  in superposition and obtain

$$|\varphi_3\rangle = \sum_{x \in W_n} |x\rangle |f(x)\rangle$$

4. Measure the second register and obtain a certain value  $z$  in the image of  $f$ . In the first register we have a whole coset  $g_0U$  of the hidden subgroup  $U$ :

$$|\varphi_4\rangle = \sum_{f(x)=z} |x\rangle |z\rangle = \sum_{x \in g_0U} |x\rangle |z\rangle.$$

(Like in the case of Simon's algorithm, this step can be omitted.)

5. Now solve the hidden subgroup problem for the normal subgroup  $N$ , which is the base group of  $W_n$ . This can be done by application of the standard algorithm for  $\mathbb{Z}_2^{2n}$  on the first  $2n$  qubits.

6. Application of the Fourier transform on the first register using the circuit given in section 5 transforms the coset into a superposition of the form  $\sum_{x \in U^\perp} \varphi_{g_0, y} |y\rangle$  in case  $g_0 \in N$  (with certain phase factors  $\varphi_{g_0, y}$  which depend on  $g_0$  and  $y$  and are from  $\{\pm 1\}$ ). If  $g_0 \in W_n \setminus N$  we get a superposition over the conjugated group  $\sum_{x \in (U^t)^\perp} \varphi_{g_0, y} |y\rangle$ ,

7. Now measure the first register. With probability  $1/2$  we draw  $g_0$  from  $N$  resp.  $W_n \setminus N$ , i. e., we get a superposition over  $U^\perp$  resp.  $(U^t)^\perp$  which leads (by performing measurements) to either equal distribution over  $U^\perp$  or equal distribution over  $(U^t)^\perp$ .

8. Iterating steps 1.–7. we generate with high probability (see lemma 6.3) the group  $U^\perp \cap N$  and the group  $(U^t)^\perp \cap N$ .

What is missing are the sets  $(U^\perp \cap N) \cdot t_0$  and  $((U^t)^\perp \cap N) \cdot t'_0$  with certain elements  $t_0$  and  $t'_0$  not in  $N$ . It is clear that it is sufficient to find only one element in one of these two sets, since then the whole group  $\langle U^\perp, (U^t)^\perp \rangle$  will be generated. But if any, there are many elements of this form in  $U^\perp$  resp.  $(U^t)^\perp$  since either there are none of them or exactly half of the elements of  $U^\perp$  resp.  $(U^t)^\perp$  is not in  $U^\perp \cap N$  resp.  $(U^t)^\perp \cap N$ . Summarizing:

After performing this experiment an expected number of  $4n$  times we generate with probability greater than  $1 - 2^{-n}$  the group  $\langle U^\perp, (U^t)^\perp \rangle$ .

9. By solving linear equations over  $\mathbb{F}_2$  it is easy to find generators for

$$(\langle U^\perp, (U^t)^\perp \rangle)^\perp = U \cap U^t.$$

After all we get generators for  $U = (U \cap N) \cdot (U \cap U^t)$ .

## 8 Conclusion and Outlook

We have presented a family of non-abelian groups for which the hidden subgroup problem can be efficiently solved on a quantum computer. The quantum algorithm is followed by a classical post-processing involving standard linear algebra over the finite field  $\mathbb{F}_2$  which can be done efficiently on a classical machine.

The groups discussed are certain wreath products  $W_n$  and our approach uses a special property of the subgroups of  $W_n$  to split the task of finding generators in two steps: First an abelian hidden subgroup problem is solved and next the non-abelian Fourier transform for  $W_n$  is used to sample from two sets which in turn allow reconstruction of the hidden subgroup.

It seems possible to generalize this result to arbitrary split extensions of the form  $\mathbb{Z}_2^n \rtimes_\varphi \mathbb{Z}_2$  and to examine an approach using representation theory instead of the pairing used in the paper.

## References

- [1] A. Barenco, Ch. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter. Elementary gates for quantum computation. *Physical Review A*, 52(5):3457–3467, November 1995. LANL e-preprint quant-ph/9503016.
- [2] Th. Beth. *Methoden der schnellen Fouriertransformation*. Teubner, 1984.
- [3] G. Brassard and P. Høyer. An Exact Polynomial-Time Algorithm for Simon's Problem. In *Proceedings of Fifth Israeli Symposium on Theory of Computing and Systems*, pages 12–33. ISTCS, IEEE Computer Society Press, 1997. LANL preprint quant-ph/9704027.
- [4] M. Ettinger and P. Høyer. On Quantum Algorithms for Noncommutative Hidden Subgroups. LANL e-preprint quant-ph/9807029, 1998.
- [5] P. Høyer. Efficient Quantum Transforms. LANL preprint quant-ph/9702028, February 1997.
- [6] B. Huppert. *Endliche Gruppen*, volume I. Springer, 1983.
- [7] N. Jacobson. *Basic Algebra II*. Freeman and Company, 1989.

- [8] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*. Cambridge University Press, 1982.
- [9] R. Jozsa. Quantum Algorithms and the Fourier Transform. *Proc. R. Soc. Lond. A*, 454:323–337, 1998.
- [10] M. Mosca and A. Ekert. The Hidden Subgroup Problem and Eigenvalue Estimation on a Quantum Computer. In *Proceedings 1st NASA International Conference on Quantum Computing & Quantum Communications*, LNCS 1509. Springer, 1998.
- [11] M. Püschel. *Konstruktive Darstellungstheorie und Algorithmengenerierung*. PhD thesis, Univ. Karlsruhe, Informatik, 1998.
- [12] M. Püschel, M. Rötteler, and Th. Beth. Fast Quantum Fourier Transforms for a Class of non-abelian Groups. LANL e-preprint quant-ph/9807064.
- [13] P. W. Shor. Algorithms for Quantum Computation: Discrete Logarithm and Factoring. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, pages 124–134. Institute of Electrical and Electronic Engineers Computer Society Press, November 1994.
- [14] D. R. Simon. On the power of quantum computation. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, pages 116–123, Los Alamitos, CA, 1994. Institute of Electrical and Electronic Engineers Computer Society Press.