Equivalence of Decoupling Schemes and Orthogonal Arrays

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Abstract

We consider the problem of switching off unwanted interactions in a given multipartite Hamiltonian. This is known to be an important primitive in quantum information processing and several schemes have been presented in the literature to achieve this task. A method to construct decoupling schemes for quantum systems of pairwise interacting qubits was introduced by M. Stollsteimer and G. Mahler and is based on orthogonal arrays. Another approach based on triples of Hadamard matrices that are closed under pointwise multiplication was proposed by D. Leung. In this paper, we show that both methods lead to the same class of decoupling schemes. Moreover, we establish a characterization of orthogonal arrays by showing that they are equivalent to decoupling schemes which allow a refinement into equidistant time-slots. Furthermore, we show that decoupling schemes for networks of higher-dimensional quantum systems with t-local Hamiltonians can be constructed from classical error-correcting codes.

1 Introduction

An important task in the study of quantum systems is to manipulate a given system Hamiltonian by applying external control operations in such a way that in effect the time-evolution of some other desired target Hamiltonian is simulated. Typically, the available control operations are restricted and furthermore the control schemes employed are required to be efficient. Hence, the number of control operations should be a polynomial function in the number of particles which are governed the system Hamiltonian. In the context of pair-interaction (also called two-local or two-body) Hamiltonians acting on n qudits a repertoire

of techniques has been developed to use any entangling Hamiltonian for universal simulation of arbitrary couplings [1, 2, 3, 4, 5, 6, 7, 8]. Here the external control operations available are given by strong pulses which are local unitary rotations applied to the individual nodes. A cornerstone of this theory is the development of *decoupling schemes* and *selective coupling schemes*. Both are pulse sequences that switch off all unwanted interactions in a given Hamiltonian. In the case of a decoupling scheme all interactions have to be switched off. In contrast the requirement for a selective coupling scheme is that all interactions except for the interaction between two fixed nodes have to be switched off. Two methods have been proposed to achieve decoupling and selective coupling of a general pair-interaction Hamiltonians in quantum systems consisting of n qubits:

Construction I This method, which was proposed by D. Leung [4], uses triples S_x, S_y, S_z of Hadamard matrices. If the rows of these matrices satisfy a compatibility condition, a sequence of pulses around the σ_x, σ_y, and σ_z-axes in the Bloch sphere can be constructed.

Construction II In the method put forward by M. Stollsteimer and G. Mahler [9] the pulses are constructed by using orthogonal arrays which are matrices which fulfill a balancedness conditions between the rows.

The purpose of this paper is to show that these constructions are equivalent, i. e., that each admissible triple of Hadamard matrices used in Construction I leads to an orthogonal array which can be used for Construction II (and vice versa). We first show this correspondence for systems consisting of n two-dimensional systems and generalize this in the sequel to higher-dimensional systems. Also the requirement that the given Hamiltonian to be a two-body Hamiltonian can be relaxed: we show that orthogonal arrays of strength t can be used to decouple any t-local Hamiltonian.

2 The Framework: Average Hamiltonian Theory

Switching off unwanted interactions is an important primitive in the approaches to render a given Hamiltonian to simulate any other Hamiltonian [1, 2, 3, 9, 4, 5, 7]. Here simulation is usually understood in a narrow sense in which the desired target Hamiltonian is approximated up to terms of quadratic and higher orders. In the following we briefly introduce the facts of this framework of average Hamiltonian theory [10, 11] which will be needed to develop the theory of decoupling schemes.

Assume that the system Hamiltonian acts on an n-fold tensor product Hilbert space $\mathcal{H} := \mathbb{C}^d \otimes \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$, where each $\mathbb{C}^d$ denotes the Hilbert space of a so-called *qudit*. Let $B := \{\sigma_\alpha \mid \alpha = 1, \ldots, d^2 - 1\}$ be a basis of traceless matrices acting on $\mathbb{C}^d$. The most general t-local Hamiltonian for a system of n coupled qudits is given by

$$H := \sum_{s=1}^{t} \sum_{(k_1,\ldots,k_s)} \sum_{\alpha_1,\ldots,\alpha_s=1}^{d^2-1} J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s}\, \sigma_\alpha^{(k_1)} \cdots \sigma_\alpha^{(k_s)}\,, \tag{1}$$

where the second sum runs over all s-tuples with (different) entries from $\{1,\ldots,n\}$ and $J_{k_1,\ldots,k_s;\alpha_1,\ldots,\alpha_s} \in \mathbb{C}$. Here and in the following we use $A^{(k)}$ to denote the operator that acts as A on the kth qubit, i. e., $A^{(k)} := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes A \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$.

In the setting discussed in this paper the only possibilities of external control are given by local unitaries on each qudit. We assume that it is possible to implement them independently. Formally, all control operations are elements of some finite subset $\mathcal{C}$ of the group $\mathcal{U}(d)^{\otimes n}$, where $\mathcal{U}(d)$ denotes the group of unitary matrices acting on $\mathbb{C}^d$. A useful approximation is to assume that all operations in $\mathcal{C}$ can be implemented arbitrarily fast ("fast control limit"). The simulation of Hamiltonians is based on the following average Hamiltonian [10] approach. Let $t_1, t_2, \ldots, t_n$ be real numbers and $V_1, V_2, \ldots, V_N \in \mathcal{C}$ be control operations. Note that letting the system evolve for some time t (in the following referred to as "wait") has the effect to apply the unitary operator $\exp(-iHt)$. Hence the sequence

$$\text{perform } V_1, \quad \text{wait } t_1, \quad \text{perform } V_2, \quad \text{wait } t_2, \quad \ldots \quad \text{perform } V_N, \quad \text{wait } t_N$$

implements the evolution $\prod_{j=1}^{N} \exp(-U_j^\dagger H U_j \, t_j)$, where $U_j = \prod_{i=1}^{j} V_j$. We say that the scheme consists of N intervals—sometimes also referred to as *time-slots*—and use the shorthand notation $(t_1, U_1; t_2, U_2; \ldots; t_N, U_N)$, where we tacitly assume that the underlying Hamiltonian H is fixed. If the times t_j are small compared to the time scale of the natural evolution according to H this gives an approximation to the average Hamiltonian

$$\bar{H} := \sum_{j=1}^{N} t_j U_j^\dagger H U_j / \tau \,,$$

where $\tau := \sum_j t_j$ is the slow down factor, i. e., the relative running time of the evolution.

We next introduce decoupling schemes which can be used to simulate the zero Hamiltonian. For this reason they are also used in dynamical suppression of decoherence in open quantum systems ("bang-bang" control), see [12, 13, 14]. A decoupling scheme is a sequence of control operations such that the resulting average Hamiltonian is the zero matrix for all system Hamiltonians of the form in eq. (1).

Recall that a unitary operator basis (also called unitary error basis [15]) is a collection of d^2 unitaries U_i that are orthogonal with respect to the inner product $\langle A|B\rangle := 1/d \operatorname{tr}(A^\dagger B)$. Bases of unitaries with this property were already studied by Schwinger [16] and recently several explicit constructions have been found [15, 17, 18].

Definition 1 (Decoupling scheme) *A decoupling scheme $D := (p_1, U_1; \ldots; p_N, U_N)$ is given by positive real numbers p_j summing up to 1 and control operations $U_j \in \mathcal{C}$ such that*

$$\sum_{j=1}^{N} p_j U_j^\dagger H U_j = 0 \tag{2}$$

for all t-body Hamiltonians acting on n qudits. We call a decoupling scheme D regular *if the lengths of the time-slots are the same, i. e., if $p_1 = p_2 = \ldots = p_N$ and in addition if the operators applied to each node form a unitary error basis.*

Note that if the system consists of one d-dimensional node only, a decoupling scheme is equivalent to a unitary operator basis. This definition includes decoupling schemes consisting of time-slots of different length. Many of the decoupling schemes considered in the literature use only time-slots of equal length [1, 3, 9, 4, 5]. However, also decoupling schemes are used in which the intervals have different lengths, most notably the famous WaHuHa sequence [19, 20, 10].

The difficulty for systems consisting of more than one node is that we still want to use a number of operations which is polynomial in the dimension d of the individual nodes as well as in the number n of nodes. To construct schemes with this property it is necessary to be able to apply selective pulses to the nodes [9]. In the following we present two constructions of schemes which achieve decoupling for any pair of nodes, i. e, these schemes can be used to decouple any pair-interaction Hamiltonian. By applying the same sequence of pulses to a fixed pair of nodes all interactions will be switched off with the exception for the bipartite system consisting of these two nodes. This in turn can be used for universal simulation.

3 Hadamard Matrices, Sign Matrices, and Phase Matrices

In the following we give a short account of the combinatorial objects underlying the construction used in [4] to obtain decoupling schemes for pair-interaction Hamiltonians acting on n qubits. The construction relies on the concept of so-called sign matrices which generalize the refocusing schemes for spin echo experiments on n qubits. The latter have been proposed in [1] and are based on Hadamard matrices. Since we will need Hadamard matrices for the subsequent constructions we briefly recall their definition.

Hadamard matrices We denote the transposed of a matrix A by A^t. A Hadamard matrix of order N is a ± 1 matrix H_N of size $N \times N$ with the property that $H_N H_N^t = N\mathbf{1}_N$. Hadamard matrices have been studied in combinatorics for a long time and several constructions have been found. We refer to [21, 22, 23] for background on and constructions of Hadamard matrices. We give some examples for Hadamard matrices of small order (here and in the following the entries ± 1 have been abbreviated to $+/-$):

$$H_2 = \begin{pmatrix} + & + \\ + & - \end{pmatrix}, \quad H_2 \otimes H_2 = \begin{pmatrix} ++++ \\ +-+- \\ ++-- \\ +--+ \end{pmatrix}, \quad A = \begin{pmatrix} -+++ \\ +-++ \\ ++-+ \\ +++- \end{pmatrix}.$$

It is known that a necessary condition for the existence of a Hadamard matrix is that either $N = 2$ or $N \equiv 0 \bmod 4$. A long-standing conjecture is whether indeed for any $N \equiv 0 \bmod 4$ a Hadamard matrix of order N exists [21]. Since $H_{2^n} := H_2 \otimes \ldots \otimes H_2$ (n tensor factors) is a Hadamard matrix, we obtain that in dimension $N = 2^n$ at least one Hadamard matrix exists.

Sign matrices A sign matrix $S_{n,N}$ of size $n \times N$ is given by the first n rows of a Hadamard matrix of order N. Hence $S_{n,N}$ is a ± 1 matrix which satisfies $S_{n,N} S_{n,N}^t = N \mathbf{1}_n$. Recall that the Schur product of two $n \times N$ matrices A and B is denoted by $C := A \circ B$ and is defined as the entry-wise product: $C_{i,j} := A_{i,j} B_{i,j}$. As an example we define the following three sign matrices S_x, S_y, S_z of size 7×8:

$$S_x := \begin{pmatrix} + - + - + - + - \\ + + - - + + - - \\ + - - + + - - + \\ + + + + - - - - \\ + - + - - + - + \\ + + - - - - + + \\ + - - + - + + - \end{pmatrix}, \; S_y := \begin{pmatrix} + - + - - + - + \\ + - + - + - + - \\ + + + + - - - - \\ + + - - + + - - \\ + - - + - + + - \\ + - + - - + - + \\ + + - - - - + + \end{pmatrix}, \; S_z := \begin{pmatrix} + + + + - - - - \\ + - - + + - - + \\ + - - + - + + - \\ + + - - - - + + \\ + + - - + + - - \\ + - + - - + - + \\ + - + - + - + - \end{pmatrix}.$$

Besides the fact that they are closed under Schur product, i. e., $S_x \circ S_y = S_z$, these matrices have another remarkable feature: all of their rows are actually rows of $H_2 \otimes H_2 \otimes H_2$. This guarantees that S_x, S_y, and S_z are sign matrices of size 7×8. As we shall see in the next section when we study criteria for decoupling, these matrices cannot be used to decouple a general pair-interaction Hamiltonian on seven qubits since all rows of all three matrices together are not orthogonal (indeed, any row appears in each of the three matrices). However, if the Hamiltonian of a seven qubit network is of the particular form where only $\sigma_x^{(k)} \otimes \sigma_x^{(\ell)}$, $\sigma_y^{(k)} \otimes \sigma_y^{(\ell)}$, and $\sigma_z^{(k)} \otimes \sigma_z^{(\ell)}$ interaction terms occur in eq. (1), these matrices can be used for decoupling and selective coupling.

In Section 4 we will present criteria for decoupling pair-interaction Hamiltonians acting on qubits and show that they are fulfilled in case we can find sign matrices S_x, S_y, and S_z which are closed under taking the Schur product and have the additional property that their rows are pairwise orthogonal. The approach [4] requires such triples S_x, S_y, and S_z of sign matrices of size $n \times N$ which are related by $S_x \circ S_y = S_z$.

Phase matrices The restriction to consider orthogonal matrices with entries ± 1 can be relaxed by allowing the entries to be more general complex phases. This gives additional flexibility for the decoupling of pair-interaction Hamiltonians acting on higher dimensional systems (qudits) and leads to the concept of phase matrices which are defined as follows: Let $k \in \mathbb{N}$ and let $\omega = \exp(2\pi i/k) \in \mathbb{C}$ be a primitive kth root of unity. Then a phase matrix $P_{n,N}$ of order k is an $n \times N$ matrix with entries in $\{1, \omega, \ldots, \omega^{k-1}\}$ such that $P_{n,N} P_{n,N}^\dagger = N \mathbf{1}_n$. Like in case of sign matrices, we are interested in collections of phase matrices which satisfy certain compatibility conditions. These conditions can be conveniently stated in terms of characters of some finite abelian group G. For the necessary background on characters of abelian groups see Appendix A. In the following we assume that the elements of G are given in a fixed order $g_1, \ldots, g_{|G|}$ and that the irreducible characters of G are in one-to-one correspondence with the elements of G and are given by $\{\chi_g : g \in G\}$, cf. Theorem 15. Recall that the exponent $e(G)$ of G is the smallest positive integer such that $g^{e(G)} = 1$ for all $g \in G$. Now, let $P_1, \ldots, P_{|G|}$ be phase matrices of order $e(G)$ which are labeled by

the elements of G. We say that the P_g, where $g \in G$, are *compatible with respect to the Schur product* if

$$P_g \circ P_h = P_{gh} \tag{3}$$

holds for all pairs $g, h \in G$. Note that according to this definition sign matrices are a special case of phase matrices. Indeed, we obtain that any sign matrix is a phase matrix for the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ where the phase matrix corresponding to the identity is given by the all-ones matrix.

The connection to decoupling schemes is as follows: the phase matrix P_h will describe the phase factors which are acquired when a fixed unitary matrix U_h is conjugated by some other matrices. The condition in eq. (3) ensures that each vector $v_{k,j} := [P_{h;k,j}]_{h \in G}$ for fixed $k = 1, \ldots, n$ and $j = 1, \ldots, N$ is a homomorphism from G to $\mathbb{C}^\times$. Since the characters form a group and the order of the elements is fixed we obtain that $v_{k,j}$ is a row from the character table of G, i. e., $v_{k,j} = \chi_g$ for an element $g \in G$. Each element $g \in G$ corresponds to a control operation U_g. These facts will be used in Section 4.2 to show the vanishing of terms in a pair-interaction Hamiltonian in case a decoupling scheme is applied.

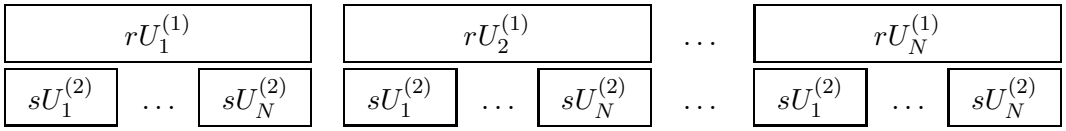
For general G it turns out to be a non-trivial task to construct $|G|$-tuples of phase matrices which at the same time fulfill condition (3). In Section 5.1 we will give a construction which is possible in case the dimension of the nodes is a prime power, i. e., $|G| = p^m$, where p is prime and $m \in \mathbb{N}$.

4 Constructing Decoupling Schemes

We continue the investigation of decoupling schemes with an observation concerning the relative lengths of the time-slots in the scheme. Recall that according to Definition 1 a scheme is regular if all intervals are of equal lengths. In Section 4.1 we show that not all decoupling schemes are regular, and that this is the case even if we are allowed to reorder and refine the time intervals. In Sections 4.2 and 4.3 we will then introduce the two constructions for schemes for decoupling and selective coupling mentioned in the introduction.

4.1 Decoupling schemes which are not regular

Assume that a decoupling scheme $D = (p_1, U_1, p_2, U_2, \ldots, p_N, U_N)$ on a system consisting of n nodes is given. Since U_j is a local operation for each $j = 1, \ldots, N$ we have that $U_j = U_j^{(1)} \otimes \ldots \otimes U_j^{(n)}$. Focusing on the first two nodes only, we can always obtain a new decoupling scheme, which has the form



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where $r, s \in \mathbb{R}$ are such that $rN = sN^2 = \sum_{j=1}^N p_j = 1$. However, in order to obtain a regular scheme in general we cannot continue in this fashion to more than three nodes. The

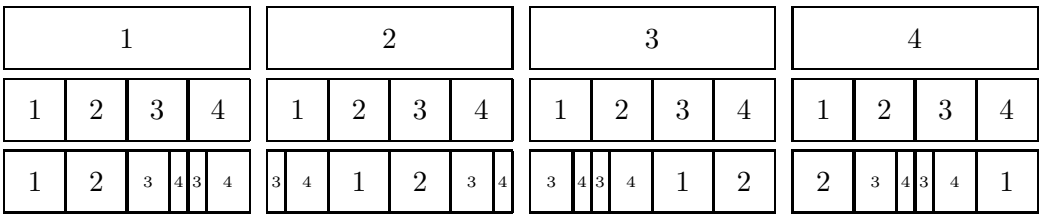

Figure 1: A decoupling scheme for a system of three qubits which is not regular, i. e., the time-slots cannot be rearranged into a form where all time-slots have the same length. The transformations applied to the individual qubits correspond to the Pauli matrices as follows: $1 = \mathbf{1}_2$, $2 = \sigma_x$, $3 = \sigma_y$, and $4 = \sigma_z$. The time intervals indicated in the figure have four different basic lengths t_1 , $t_2 = 1/4\, t_1$, $t_3 = \frac{1}{\sqrt{2}}\, t_2$, and $t_4 = \frac{\sqrt{2}-1}{\sqrt{2}}\, t_2$. For instance, the first intervals applied to the first qubit all length t_1, whereas the interval lengths for qubit three are given by t_2, t_2, t_3, t_4, t_4, t_3, etc.

pulse sequence given in Figure 1 provides an example of a decoupling scheme that cannot be refined into time-slots which have the same lengths. The control operations used in the scheme are the Pauli matrices, which form a basis for the vector space of all 2×2 matrices and are given by

$$\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the given example the network consists of three qubits and in each time-slot precisely one of the four Pauli matrices is applied. Indeed, we first verify that the pulse sequence defined in Figure 1 defines a decoupling scheme for any pair-interaction Hamiltonian: first, note that the sum of the times for each Pauli operator applied to the individual qubits is constant, i. e., the local terms are removed. Moreover, by considering pairs of rows we verify directly that also any pair of symbols (a, b) with $a, b \in \{1, 2, 3, 4\}$ is applied for the same time t_2. For example in case of rows two and three we obtain for the pairs $(1, 1)$ the total time t_2 and for $(3, 4)$ the total time $t_3 + t_4 = t_2$. However, the sequence cannot be subdivided into a finite number of intervals of equal lengths. Indeed, this would contradict the fact that $\sqrt{2}$ is not a rational number.

4.2 Decoupling schemes from sign and phase matrices

We now describe the approach of [4] to construct decoupling and selective coupling schemes for general pair-interaction Hamiltonians acting on qubits. Then we present a generalization for qudits by generalizing the underlying group-theoretical structures based on our definition of phase matrices.

The qubit case: decoupling schemes from sign matrices A general pair-interaction Hamiltonian, i. e., two-local Hamiltonian, for n qubits may be written in the form

$$H := \sum_{k<\ell} \sum_{\alpha\beta} J_{k\ell;\alpha\beta} \sigma_\alpha^{(k)} \sigma_\beta^{(\ell)} + \sum_k \sum_\alpha J_{k;\alpha} \sigma_\alpha^{(k)} \tag{4}$$

where $J_{k\ell;\alpha\beta} \in \mathbb{R}$, $J_{k;\alpha} \in \mathbb{R}$ and where σ_α are the Pauli matrices, i. e., $\alpha \in \{x, y, z\}$. To construct decoupling schemes we choose $\mathbf{1}_2$, σ_x, σ_y and σ_z as control operations. Then in each time-slot of a decoupling scheme each of the n qubits is conjugated by precisely one of the matrices $\mathbf{1}_2$, σ_x, σ_y, and σ_z. Since the terms of the Hamiltonian in eq. (4) are expressed in terms of the Pauli matrices we have to compute the resulting effect by this local conjugation: the possible sign assignments for $\mathbf{1}_2^{(k)}, \sigma_x^{(k)}, \sigma_y^{(k)}, \sigma_z^{(k)}$ are given by the following table:

	$\mathbf{1}_2$	σ_x	σ_y	σ_z
$\mathbf{1}_2$	$+$	$+$	$+$	$+$
σ_x	$+$	$+$	$-$	$-$
σ_y	$+$	$-$	$+$	$-$
σ_z	$+$	$-$	$-$	$+$

(5)

Here the rows are labeled by the operators used in eq. (4) and the columns by the operators realizing the conjugation at each time-slot. Note that in each column the signs multiply to $+1$ since conjugation by a local unitary corresponds to a $SO(3)$ rotation of the Bloch vector. Hence, the signs assignments of the three Pauli matrices $\sigma_\alpha^{(k)}$ acting on the same qubit k are not independent but rather the third is always given by the other two.

Now, in each interval $\sigma_\alpha^{(k)}$ acquires either a $+$ or a $-$ sign, which is controlled by the applied local unitaries (the identity matrix or a Pauli matrix). According to table (5) the bilinear coupling $J_{k\ell;\alpha\beta}\sigma_\alpha^{(k)}\sigma_\beta^{(\ell)}$ is unchanged (negated) when the signs of $\sigma_\alpha^{(k)}$ or $\sigma_\alpha^{(\ell)}$ agree (disagree).

Decoupling criteria in terms of sign matrices We show that sign matrices satisfying certain orthogonality conditions yield decoupling schemes. The (k, j) entry of S_α for $\alpha \in \{x, y, z\}$ is denoted by $S_{\alpha;kj}$ and gives the sign of $\sigma_\alpha^{(k)}$ in the jth time-slot. Hence a regular decoupling scheme which uses N time-slots can be obtained from these matrices using the following rules: if the triple of entries at position (k, j) is given by $(+++)$,$(+--)$,$(-+-)$, respectively $(--+)$ then the operation applied to qubit k in time step j is given by $\mathbf{1}_2$, σ_x, σ_y, respectively σ_z.

Decoupling is achieved if any two rows taken from S_x, S_y, S_z are orthogonal. To achieve selective coupling between two nodes k and ℓ, the operations applied to the nodes k and ℓ are chosen to be identical while still maintaining orthogonality of the modified sign matrices [4]. Within this framework sufficient and necessary conditions for decoupling of all pair-

interactions are given by the following equations:

$$\sum_{j=1}^{N} S_{\alpha;kj} = 0 \tag{6}$$

for all α and all k, and

$$\sum_{j=1}^{N} S_{\alpha;kj} S_{\beta;\ell j} = 0 \tag{7}$$

for all α, β and all $k < \ell$. The first condition ensures that all local terms are removed and the second condition that all bilinear terms are removed. These conditions are satisfied if the sign matrices S_x, S_y, S_z and all rows of all three matrices are orthogonal to each other.

Example 2 As an example consider the following sign matrices which specify a decoupling scheme for a system of 5 qubits with 16 time-slots.

$$S_x := \begin{pmatrix} + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \end{pmatrix}, \tag{8}$$

$$S_y := \begin{pmatrix} + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \end{pmatrix}, \tag{9}$$

$$S_z := \begin{pmatrix} + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \\ + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \end{pmatrix}. \tag{10}$$

In general the construction of admissible triples (S_x, S_y, S_z) of sign matrices proves to be a delicate task. However, a construction of triples of sign matrices of size $(2^{2n}-1)/3 \times 2^{2n}$ by partitioning the rows of the Hadamard matrices $H_2^{\otimes 2n}$, where $n \in \mathbb{N}$, was given in [4]. This construction has been revisited in [5] where an alternative proof based on spreads in a finite geometry has been given. We will give yet another proof of this family of sign matrices in Section 5.1 which is based on Hamming codes.

Generalization to the qudit case: phase matrices In the following we generalize the approach described in [4] to pair-interactions between higher-dimensional systems, i. e., qudits. It will be useful to express a general pair-interaction Hamiltonian with respect to a so-called nice error basis and to use the matrices from such basis as control operations. First, we recall the definition of nice error bases [15, 24, 25].

Definition 3 (Nice error basis) *Let G be a group of order d^2 with identity element e. A* nice error basis *on $\mathbb{C}^d$ is a set $\mathcal{E} = \{U_g \in \mathbb{C}^{d\times d} \mid g \in G\}$ of unitary matrices, which are labeled by the elements of G, such that (i) U_e is the identity matrix, (ii) $\operatorname{tr} U_g = d\,\delta_{g,e}$ for all $g \in G$, and (iii) $U_g U_h = \alpha(g,h) U_{gh}$ for all $g, h \in G$. The factor system $\alpha(g,h)$ is a function from $G \times G$ to the set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.*

Condition (ii) shows that the matrices U_g are pairwise orthogonal with respect to the trace inner product. The group G is called *index group* since its group elements index the elements of the nice error basis $\mathcal{E}$.

Example 4 *Let $d \in \mathbb{N}$ and let $\omega = \exp(2\pi i/d)$ denote a primitive d-th root of unity. Next, we define operators $S := \sum_{k=0}^{d-1} |k\rangle\langle k+1|$, where the indices are reduced modulo d, and $T := \sum_{k=0}^{d-1} \omega^k |k\rangle\langle k|$. Then the set $\mathcal{E}_d := \{S^i T^j : i = 0, \ldots, d-1, j = 0, \ldots, d-1\}$ is a nice error basis on $\mathbb{C}^d$ (see, e. g., [26]). This shows the existence of nice error bases for any dimension $d \in \mathbb{N}$. In this case the index group is the abelian group $G = \mathbb{Z}_d \times \mathbb{Z}_d$. The identity $ST = \omega TS$ is readily verified. This shows that the corresponding factor system α is given by $\alpha((i,j),(k,\ell)) = \omega^{-jk}$, for all $(i,j),(k,\ell) \in G$.*

Next, we describe a particular way of representing a general pair-interaction Hamiltonian acting on n qudits. Let $\mathcal{E} = \{U_g : g \in G\}$ be a nice error basis. Since the matrices U_g form a basis of $\mathbb{C}^{d\times d}$ a general pair-interaction Hamiltonian on n coupled qudits may be written as

$$H := \sum_{k<\ell} \sum_{h,h' \neq e} J_{k\ell;hh'} U_h^{(k)} U_{h'}^{(\ell)} + \sum_k \sum_{h \neq e} J_{k;h} U_h^{(k)} \,, \tag{11}$$

where the coefficients $J_{k\ell;hh'} \in \mathbb{C}$ and $J_{k;h} \in \mathbb{C}$ are chosen such that H is a traceless Hermitian matrix. In the following we consider decoupling schemes with time-slots of equal length and with elements of a nice error basis having abelian index groups as control operations. Then, in each time-slot, each U_h in eq. (11) acquires a phase factor that is controlled by the applied local unitaries of the nice error basis. We define $\chi(g,h)$ to be the phase factor that U_h acquires when it is conjugated by U_g, i. e. $\chi(g,h)$ is defined via the relation

$$U_g^\dagger U_h U_g = \chi(g,h) U_h \,. \tag{12}$$

Hence, the bilinear term $U_h \otimes U_{h'}$ acquires the phase factor $\chi(g,h)\chi(g',h')$ if it is conjugated by $U_g \otimes U_{g'}$. The fact that conjugation by U_g of U_h merely introduces a phase factor follows from the index group being abelian. We need to characterize the corresponding $d^2 \times d^2$ matrix

$$\mathcal{X} := (\chi(g,h))_{g,h\in G} \tag{13}$$

with entries $\chi(g,h)$.

Lemma 5 *Let $\mathcal{E} := \{U_g \mid g \in G\}$ be a nice error basis with an abelian index group G. Then the matrix $\mathcal{X} = (\chi(g,h))_{g,h\in G}$ is the character table of the group G.*

We give a proof in Appendix B. This lemma shows that $\mathcal{X}$ is a character table of G and thereby provides a natural generalization of Table 5 to non-qubit systems. Given a decoupling scheme which contains operators from a nice error basis $\mathcal{E} = \{U_g : g \in G\}$ we obtain a collection of phase matrices P_h for each $h \in G$ in the following way: the (k, j)th entry of P_h—denoted by $P_{h;k,j}$—is given by the phase factor of U_h that is acquired in the jth time-slot by conjugating the k qudit by the respective element of the scheme.

Criteria for decoupling in term of phase matrices Let P_h be phase matrices. Then they define a decoupling scheme if and only if

$$\sum_{j=1}^{N} P_{h;kj} = 0 \tag{14}$$

for all k and for all $h \neq e$, and

$$\sum_{j=1}^{N} P_{h;kj} P_{h';\ell j} = 0 \tag{15}$$

for all $k < \ell$ and for all $h, h' \neq e$. The condition given in eq. (14) ensures that all local terms are removed. Eq. (15) implies that all bilinear terms are removed. The question how to construct collections of phase matrices which satisfy eqs. (14) and (15) is addressed in Section 5.1 and a solution is presented if all nodes have a dimension which is a power of a prime.

4.3 Decoupling schemes from orthogonal arrays

Orthogonal arrays have been applied in the design of experiments to plan statistical data collections systematically. The books [21, 23, 27] provide good introductions to the topic. In the following we recall the definition of orthogonal arrays (or OAs for short) and show how to use them for the decoupling problem. M. Stollsteimer and G. Mahler have first used OAs for the construction of decoupling schemes and selective coupling schemes [9] for pair-interaction Hamiltonians acting on qubits. This method was generalized to pair-interaction Hamiltonians acting on qudit [5] with the helps of unitary error bases. Even more generally, we show in this section how to use orthogonal arrays for constructing decoupling schemes for t-local Hamiltonians acting on qudits with $t > 2$.

Definition 6 (Orthogonal array of strength t) *Let $\mathcal{A}$ be a finite set and let $n, N \in \mathbb{N}$. An $n \times N$ array M with entries from $\mathcal{A}$ is an orthogonal array with $|\mathcal{A}|$ levels, strength t, and index λ if and only if every $t \times N$ sub-array of M contains each possible t-tuple of elements in $\mathcal{A}$ precisely λ times as a column. We use the notation $OA_\lambda(N, n, s, t)$ to denote a corresponding orthogonal array. If λ, s, and t are understood we also use the shorthand notation $OA(N, n)$.*

In statistics for the various parameters of an orthogonal array (OA) some traditional terminology is used. In the context of decoupling and simulation of Hamiltonians different

Parameter	Design of Experiments	Decoupling Schemes for qudit systems
n	factors	nodes (qudits)
N	runs	time-slots
$\mathcal{A}$	levels	elements of an operator basis
d	number of levels	(dimension of the nodes)^2
t	strength	locality
λ	index	—

Table 1: Dictionary between notions used in the theory of design of experiments and the theory of qudit systems to describe the parameters of an orthogonal array $OA_\lambda(N, n, d, t)$ over alphabet $\mathcal{A}$.

names for these parameters are used. We provide a dictionary between the different languages in table 1. Note that as a convention we write OAs as $n \times N$ matrices, whereas most authors in design theory prefer to write the matrices as $N \times n$ matrices. Besides typographic reasons we found the presentation using $n \times N$ matrices useful to establish the correspondence with pulse sequences in NMR which are typically read from left to right like a musical score [10].

An important special case arises if the strength t is two. This means that each pair of elements of $\mathcal{A}$ occurs λ times in the list $((a_{kj}, a_{\ell j}) \mid j = 1, \ldots N)$ for $1 \leq k < \ell \leq n$. Most of the known constructions actually yield arrays of strength two [27]. For many physical systems it will be sufficient to study arrays of small strength since the strength relates to the degree of the interactions, i. e., for pair-interaction Hamiltonians it is sufficient to consider arrays of strength $t = 2$.

Example 7 As an example of small size we give an orthogonal array with parameters $OA(16, 5, 4, 2)$. This means that we have 16 runs/time-slots, 5 factors/qubits, 4 different symbols/pulses, and (pair-interaction) strength two.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\
1 & 3 & 4 & 2 & 1 & 3 & 4 & 2 & 3 & 1 & 2 & 4 & 3 & 1 & 2 & 4 \\
1 & 3 & 4 & 2 & 4 & 2 & 1 & 3 & 4 & 2 & 1 & 3 & 1 & 3 & 4 & 2 \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 4 & 3 & 2 & 1 & 3 & 4 & 1 & 2
\end{pmatrix}$$

It is straightforward to check that indeed every pair of rows contains all the 16 possible pairs of symbols precisely once. This array was obtained from a linear error-correcting code over the finite field $\mathbb{F}_4$. We will explore this construction in more detail in Theorem 10.

Next, we generalize the ideas in [9, 5] and describe how to achieve decoupling of t-local Hamiltonian acting on qudits. The basic idea is to use an orthogonal array M with parameters $OA(N, n, d^2, t)$ over an alphabet $\mathcal{A}$ of size d^2. Here d denotes the dimension of the qudits. The elements of $\mathcal{A}$ are identified with the operators $U_1, \ldots, U_{d^2}$ of a unitary

error basis [5]. The orthogonal array is an $n \times N$ matrix $M = (m_{kj})$ and determines which elements of the unitary error basis are used on the qudits in the time-slots as follows:

$$\frac{1}{N} \sum_{j=1}^{N} (U_{m_{1j}}^{\dagger} \otimes \cdots \otimes U_{m_{nj}}^{\dagger}) H (U_{m_{1j}} \otimes \cdots \otimes U_{m_{nj}}). \tag{16}$$

The following theorem shows that the resulting time evolution is that of the zero Hamiltonian which means that indeed the decoupling conditions given in Definition 1 are satisfied.

Theorem 8 (Decoupling with OAs) *Any orthogonal array $OA(N, n, d^2, t)$ over an alphabet of size d^2 can be used to decouple n qudits which are governed by a t-local Hamiltonian within N time-slots.*

Proof. First, note that for any t-local Hamiltonian H of a system consisting of t qudits the following operations define a decoupling scheme [5]: Let $U_1, U_2, \ldots, U_{d^2}$ denote the elements of some unitary error basis for $\mathbb{C}^d$. Since the tensor products of all possible pairs of these elements form a vector space basis of the linear maps acting on $(\mathbb{C}^d)^{\otimes t}$ we obtain that [17, 5]

$$\frac{1}{d^{2t}} \sum_{i_1,\ldots,i_t=1}^{d^2} (U_{i_1}^{\dagger} \otimes \ldots \otimes U_{i_t}^{\dagger}) H (U_{i_1} \otimes \ldots \otimes U_{i_t}) = 0 \tag{17}$$

for all (traceless) Hamiltonians acting on $(\mathbb{C}^d)^{\otimes t}$. Recall that we assume without loss of generality that H is traceless. Let $B := \{\sigma_1, \sigma_2, \ldots, \sigma_{d^2-1}\}$ be a basis for the vector space of traceless matrices of size $d \times d$. Recall that a general t-local Hamiltonian acting on n qudits can be written as

$$H = \sum_{s=1}^{t} \sum_{(k_1,\ldots,k_s)} \sum_{\alpha_1,\ldots,\alpha_s=1}^{d^2-1} J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \, \sigma_{\alpha_1}^{(k_1)} \ldots \sigma_{\alpha_s}^{(k_s)} \tag{18}$$

where the second sum runs over all s-tuples with different entries from $\{1, \ldots, n\}$ and $J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \in \mathbb{C}$. Now, we pick any s-subset $\{k_1, \ldots, k_s\} \subseteq \{1, \ldots, n\}$ of the nodes and denote by $C_{k_1,\ldots,k_s}$ the coupling between these nodes. We define $C_{k_1,\ldots,k_s}$ to be the coupling among the qudits $k_1, \ldots, k_s$, i. e.,

$$C_{k_1,\ldots,k_s} := \sum_{\alpha_1,\ldots,\alpha_s} J_{(k_1,\ldots,k_s);\alpha_1,\ldots,\alpha_s} \, \sigma_{\alpha_1}^{(k_1)} \otimes \ldots \otimes \sigma_{\alpha_s}^{(k_s)} \, .$$

We define $\hat{C}_{k_1,\ldots,k_s}$ to be the corresponding operator acting on $(\mathbb{C}^d)^{\otimes s}$. Formally, we say that $C_{k_1,\ldots,k_s}$ is obtained by embedding $\hat{C}_{k_1,\ldots,k_s}$ into $(\mathbb{C}^d)^{\otimes n}$ according to the tuple $(k_1, \ldots, k_s)$. For any operator X acting on $(\mathbb{C}^d)^{\otimes s}$ we denote the embedding into $(\mathbb{C}^d)^{\otimes n}$ according to the tuple $(k_1, \ldots, k_s)$ by $X^{(k_1,\ldots,k_s)}$.

The idea of the proof is to reduce the problem to eq. (17) by using the local structure of the Hamiltonian. Since M is an $OA(N, n, d^2, t)$ all elements of $\{1, 2, \ldots, d^2\}^s$ for $s \leq t$

appear equally often in the list $(m_{k_1,j},\ldots,m_{k_s,j})$ where $j=1,\ldots,N$. Therefore, the average Hamiltonian corresponding to the coupling among the qudits $k_1,\ldots,k_s$ is evaluated as follows:

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}(U_{m_{1j}}^{\dagger}\otimes\cdots\otimes U_{m_{nj}}^{\dagger})\,C_{k_1,\ldots,k_s}\,(U_{m_{1j}}\otimes\cdots\otimes U_{m_{nj}})\\
=&\quad\left[\frac{1}{N}\sum_{j=1}^{N}(U_{m_{k_1,j}}^{\dagger}\otimes\cdots\otimes U_{m_{k_s,j}}^{\dagger})\,\hat{C}_{k_1,\ldots,k_s}\,(U_{m_{k_1,j}}\otimes\cdots\otimes U_{m_{k_s,j}})\right]^{(k_1,\ldots,k_s)}\\
=&\quad\left[\frac{1}{d^{2t}}\sum_{i_1,\ldots,i_s=1}^{d^2}(U_{i_1}^{\dagger}\otimes\ldots\otimes U_{i_s}^{\dagger})\,\hat{C}_{k_1,\ldots k_s}\,(U_{i_1}\otimes\ldots\otimes U_{i_s})\right]^{(k_1,\ldots,k_s)}=0\,.
\end{aligned}$$

The equality between the second last and last line is due to eq. (17). □

5 Equivalence of the Constructions

We show that the methods based on phase matrices and orthogonal arrays of strength two lead to the same class of decoupling schemes if we use elements of a nice error basis with an abelian index group as control operations. More precisely, we prove that the decoupling conditions given in eqs. (14) and (15) are equivalent to the condition that the decoupling matrix is an orthogonal array of strength two.

5.1 Phase matrices from orthogonal arrays

We show that a decoupling scheme based on an orthogonal arrays of strength two gives rise to phase matrices satisfying the decoupling criteria eqs. (14) and (15) for pair-interaction (two-local) Hamiltonians.

Theorem 9 *Let G be a finite abelian group of exponent $e(G)$ and let $\mathcal{E}=\{U_g \mid g\in G\}$ be a nice error basis for $\mathbb{C}^d$ with index group G. Then a decoupling scheme for n qudits governed by a pair-interaction Hamiltonian can be constructed. The scheme uses N time-slots and is constructed from an orthogonal array $OA(N,n,d^2,2)$ over the alphabet $\mathcal{A}=\{1,2,\ldots d^2\}$. Furthermore, the scheme gives rise to phase matrices $[P_h]_{h\in G}$ of size $n\times N$ with entries in $\{1,\omega,\ldots,\omega^{e(G)-1}\}$ which satisfy the orthogonality conditions and are compatible with respect to taking Schur products.*

Proof. We denote the orthogonal array by $M=[m_{k,j}]$, where $k=1,\ldots,n$ and $j=1,\ldots,N$. Next, we fix an ordering $g_1,\ldots,g_{d^2}$ of the elements of G and assume that $g_1=e$ is the identity. Moreover, we identify the operators of $\mathcal{E}$ with the elements of $\mathcal{A}$ according to $1\mapsto g_1$, $2\mapsto g_2,\ldots,d^2\mapsto g_{d^2}$. Note that conjugating $U_h\in\mathcal{E}$ by $U_g\in\mathcal{E}$ results in a phase factor that is the (g,h)th entry of the character table of G.

Starting from the given orthogonal array we construct d^2 phase matrices $P_{g_1}, \ldots, P_{g_{d^2}}$ as follows. Pick row number k of the OA and replace each symbol according to $a \mapsto v_{g_a}$ for $a = 1, \ldots d^2$ where v_{g_a} denotes the g_ath row of the character table of G (cf. Lemma 5). By assigning P_{g_i} for $i = 1, \ldots, d^2$ the first, second, and d^2th components of each entry, we define the rows number k of the d^2 phase matrices. In other words, the entry (k, j) of the matrix P_h, where $h \in \{g_1, \ldots, g_{d^2}\}$, is given by $P_{h;k,j} = \chi(m_{k,j}, h)$. Note that the matrix P_{g_1} is the all-ones matrix of size $n \times N$.

While the condition $P_g \circ P_h = P_{gh}$ is automatically guaranteed since the characters form a group, we have to show that the resulting vectors are pairwise orthogonal. In order to do so we pick two rows k and ℓ of the original orthogonal array. We may assume that the two rows have the following form (or else we apply a suitable permutation of the columns)

$$\underbrace{\left(\begin{array}{cccc|cccc|c|cccc} 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 2 & \ldots & d^2 & d^2 & \ldots & d^2 \\ 1 & 2 & \ldots & d^2 & 1 & 2 & \ldots & d^2 & \ldots & 1 & 2 & \ldots & d^2 \end{array}\right)}_{\lambda \text{ times}} \tag{19}$$

since all pairs appear equally often (λ times) in the OA. Let $\vec{\lambda} = (+ + \cdots +)$ be the vector of length $\lambda = N/d^4$ containing only the entry $+1$. Furthermore, for each $h \in \{g_1, \ldots, g_{d^2}\}$ define a vector $w_h \in \mathbb{C}^{d^2}$ as follows. We define $w_h := [\chi(g_1, h), \ldots, \chi(g_{d^2}, h)]$, i. e., w_h is the hth column of the character table of G (cf. Lemma 5). By substituting the entries of rows k and ℓ in the form of eq. (19) by the corresponding sign assignments in Table (5) we obtain as the kth rows of $P_{g_1}, \ldots, P_{g_{d^2}}$ the vectors

$$\vec{\lambda} \otimes w_{g_1} \otimes (+ \cdots +), \quad \vec{\lambda} \otimes w_{g_2} \otimes (+ \cdots +), \quad \ldots, \quad \vec{\lambda} \otimes w_{g_{d^2}} \otimes (+ \cdots +)$$

and for the ℓth rows of $P_{g_1}, \ldots, P_{g_{d^2}}$ the following vectors:

$$\vec{\lambda} \otimes (+ \cdots +) \otimes w_{g_1}, \quad \vec{\lambda} \otimes (+ \cdots +) \otimes w_{g_2}, \quad \ldots, \quad \vec{\lambda} \otimes (+ \cdots +) \otimes w_{g_{d^2}},$$

where $(+ \cdots +)$ is a vector of length d^2. Whenever g_i, g_j are not both equal to the identity g_1 all these vectors are orthogonal to each other since the columns of the character table are orthogonal. This shows that all rows of the matrices $P_{g_2}, \ldots, P_{g_{d^2}}$ are orthogonal and the matrices satisfy the Schur condition $P_g \circ P_h = P_{gh}$. □

Phase matrices from Hamming codes In the following we show how the known constructions of sign matrices can be reproduced with well known families of orthogonal arrays. What is more, we show that the class of orthogonal arrays used for this construction are based on Hamming codes [28, 29]. They can be used to construct phase matrices for higher-dimensional systems in case the dimension d of the qudits is a power of a prime.

First, we briefly recall some basic facts about error-correcting codes [28, 29] since they will feature in the subsequent constructions of orthogonal arrays. A linear code over the finite field $\mathbb{F}_q$ is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$. The metric on the space $\mathbb{F}_q^n$ is called the Hamming weight. For $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ we have that $\mathrm{wt}(x) :=$

$|\{i \in \{1,\ldots,n\} : x_i \neq 0\}|$. The minimum distance of a linear code C is defined by $d = d_{\min} := \min\{\mathrm{wt}(c) : c \in C, c \neq 0\}$, where 0 denotes the zero vector. As a shorthand we often abbreviate this situation by saying that C is an $[n,k,d]_q$ code. We need one more definition which is the dual code $C^\perp$ of C defined by $C^\perp := \{x \in \mathbb{F}_q^n : x \cdot y = 0 \,\text{for all}\, y \in C\}$.

The following theorem [27, Theorem 4.6] establishes a connection between orthogonal arrays and error-correcting codes. In fact this is one of the most prolific constructions for OAs known.

Theorem 10 (OAs from linear codes) *Let C be a linear $[n,k,d]_q$ code over $\mathbb{F}_q$. Let $d^\perp$ be the minimum distance of the dual code $C^\perp$. Arrange the code words of C into the columns of a matrix $A \in \mathbb{F}_q^{n \times q^k}$. Then A is an $OA(q^k, n, q, d^\perp - 1)$.*

For the case of a network consisting of n qubits which are governed by a pair-interaction Hamiltonian we can construct decoupling schemes using N pulses from an $OA(N,n,4,2)$. Hence, in order to apply Theorem 10 we have to find a code C of linear codes over $\mathbb{F}_4$ for which the parameters are $[n,k,d]$ and for which the minimum distance $d^\perp$ of the dual code is at least 3.

Let q be a prime power and let $m \in \mathbb{N}$. Then the Hamming code $H_{q,m}$ of length $n = (q^m-1)/(q-1)$ is a single-error correcting code over the field $\mathbb{F}_q$ with parameters $[n, n-m, 3]_q$. The dual code $H_{q,m}^\perp$ of the Hamming code $[n, n-m, 3]_q$ has parameters $[n, m, q^{m-1}]$. By specializing $q = 4$ and by using Theorem 10 for $H_{4,m}^\perp$ we therefore obtain orthogonal arrays with parameters $OA(N,n,4,2)$, where $n = (4^m-1)/3$ and $N = 4^m$ for any choice of $m \in \mathbb{N}$. The alphabet set is in this case the finite field $\mathbb{F}_4$ of four elements.

The procedure to obtain a decoupling scheme for a network of n qubits, where n is an arbitrary natural number, i. e., not necessarily of the form $n = (4^m-1)/3$ is as follows: first let $m \in \mathbb{N}$ be the unique integer such that $n \leq \frac{4^m-1}{3} \leq 4n$. Then construct the orthogonal array with parameters $OA(4^m, (4^m-1)/3, 4, 2)$. The columns of this OA are code words of $H_{4,m}^\perp \subseteq \mathbb{F}_4^{(4^m-1)/3}$. We can now obtain a triple of sign matrices S_x, S_y, and S_z by using the substitution rules in Theorem 9. This leads to the same sign matrices as the ones constructed in [4] by a direct construction and in [5] using spreads in the geometry $\mathbb{F}_2^{2m}$.

In case the dimension is an arbitrary power of a prime $d = p^r$, we use the Hamming code $[n, n-m, 3]_{d^2}$ to obtain an $OA(N, n, d^2, 2)$, where $n = (q^m-1)/(q-1)$ and $N = q^m$. By Theorem 9 from this orthogonal array a collection of phase matrices can be constructed which satisfy the orthogonality conditions and are compatible with respect to taking Schur products.

5.2 Orthogonal arrays from phase matrices

In this section we provide a converse to the previous section by showing that orthogonal arrays of strength two can be constructed from phase matrices satisfying the decoupling conditions in eqs. (14) and (15). To do this we need the following lemma which gives a criterion in terms of group characters to decide whether an element of the group ring is an equally-weighted sum of all group elements. This allows to check whether a matrix is an orthogonal array.

Lemma 11 *Let G be an abelian group of order $|G|$. Denote by $\chi_1, \chi_2, \ldots, \chi_{|G|}$ all irreducible characters of G, where χ_1 is the trivial character (i. e. $\chi_1(h) = 1$ for all $h \in G$). Let v be an arbitrary element of the group ring $\mathbb{C}[G]$, i. e., v is a formal sum of (weighted) group elements*

$$v := \sum_{g \in G} \mu_g g \,, \quad \mu_g \in \mathbb{C} \,. \tag{20}$$

If $\chi_i(v) = 0$ for all $i = 2, \ldots, |G|$ then we have $v = \frac{\mu}{|G|} \sum_{g \in G} g$, where $\mu := \chi_1(v) = \sum_{g \in G} \mu_g$.

A proof of this lemma is given in Appendix B. We are now ready to state the main result of this section.

Theorem 12 *Let G be a finite abelian group and let $[P_g]_{g \in G}$ be a collection of phase matrices of size $n \times N$ which are compatible with respect to taking Schur products and satisfy the orthogonality relations. Then these phase matrices define an orthogonal array $OA(N, n, |G|, 2)$.*

Proof. For fixed $k = 1, \ldots, n$ and $j = 1, \ldots, N$ each vector $v_{k,j} := [P_h; k, j]_{h \in G}$ is a row the character table of G. Therefore, it determines uniquely $g \in G$ such that the entries of $v_{k,j}$ are $\chi_g(h)$, i. e., the values of the irreducible character corresponding to g applied to $h \in G$. We denote the so defined group element by $g_{k,j}$. Let $M = (g_{k,j})$ be the $n \times N$ matrix with entries $g_{k,j}$. We will show that M is an orthogonal array $OA(N, n, |G|, 2)$.

Pick any two rows (g_{kj}) and $(g_{\ell j})$ of M. We define an element of the group ring $\mathbb{C}[G \times G]$ as the formal sum

$$r_{k\ell} := \sum_{j=1}^{N} (g_{kj}, g_{\ell j}) \,.$$

To abbreviate the notation we denote by $\chi_{g,g'}$ the irreducible character of $G \times G$ corresponding to the element (g, g'). The decoupling conditions given in eqs. (14) and (15) are equivalent to

$$\chi_{g,g'}(r_{k\ell}) = 0$$

for all $(g, g') \neq (e, e)$. By Lemma 11 this is equivalent to the case that all elements of $G \times G$ appear equally often in the sum $r_{k\ell}$. This shows that M is an orthogonal array $OA(N, n, |G|, 2)$ of strength $t = 2$ over G. □

Theorems 9 and 12 shows that phase matrices can be used to define an orthogonal array of strength two and vice versa. Based on the above lemma we give an alternative characterization of orthogonal arrays of arbitrary strength which is a generalization of [27, Theorem 3.30]. This theorem implies that the entries of the array can be replaced by complex numbers such that the resulting matrix is orthogonal with respect to the usual inner product for strength $t = 2$. Recall that for elements of the Cartesian product $v \in G^n$ the Hamming weight $\mathrm{wt}(v)$ is defined by the number of components which are different from the identity.

Theorem 13 (Conditions for a matrix to be an OA) *Let G be a finite abelian group and let A be a matrix of size $n \times N$ with entries from G. Then A is an orthogonal array $OA(N, n, |G|, t)$ if and only if*

$$\sum_{j=1}^{N} \prod_{i=1}^{n} \chi(A_{i,j} v_i) = 0 \tag{21}$$

holds for all nontrivial characters $\chi \in \mathrm{Irr}(G)$ and for all $v \in G^n$ of Hamming weight $\mathrm{wt}(v) \leq t$.

Proof. Suppose A is an orthogonal array $OA(N, n, |G|, t)$ and let $v \in G^n$ be a fixed element of weight $\mathrm{wt}(v) \leq t$. Let ϑ be a non-trivial character of the t-fold direct product G^t. Denote by $v_t \in G^t$ the vector containing the components of v which are different from the identity. We obtain that $\sum_{g \in G^t} \vartheta(g v_t) = \sum_{g \in G^t} \vartheta(g) = 0$ and the statement follows from the fact that the characters of G^t are given by products of t characters of G.

Conversely, assume that we are given a matrix A such that eq. (21) holds for all non-trivial characters and all $v \in G^n$ of weight $\mathrm{wt}(v) \leq t$. In particular, this means that eq. (21) is satisfied for the vector w_e all components of which are equal to the identity $e \in G$. Again, fix a t-subset T of the rows of A. We have that for all non-trivial characters ϑ of G^t the identity $\sum_{j=1}^{N} \vartheta(g_j w_e) = 0$ holds, where the elements $g_j \in G^t$ are obtained by selecting the jth column of A, followed by selecting the components corresponding to T, and finally to consider the element as being an element of G^t. Now, we can apply Lemma 11 to obtain that the list $[g_j : j = 1, \ldots, N]$ has to contain all elements G^t and that each element has to occur the same number of times. This shows that A is an $OA(N, n, |G|, t)$. □

5.3 A new characterization of orthogonal arrays

In Section 4.3 we have seen that orthogonal arrays of strength t can be used to construct decoupling schemes for t-local Hamiltonians. In order to establish a converse result we need some additional conditions on the class of schemes considered: (i) the schemes have to be regular (see Definition 1) and moreover we will assume that (ii) the pulses are actually taken from a fixed set of unitaries which in addition will be assumed to form a unitary error basis. We begin by stating some standard concepts from quantum information theory which will be used in the proof. Recall that the *Shannon entropy* is defined by the equation

$$H(p_1, \ldots, p_M) = -\sum_{j=1}^{M} p_j \log_2 p_j \,.$$

Shannon entropy measures the disorder of probability distributions. If $p_j = 1$ for some j, then the entropy is zero. The entropy takes its maximum value $\log_2 M$ for the uniform distribution. The notion of entropy extends to density operators, and is usually called *von Neumann entropy*. Let ρ be an arbitrary density operator on $\mathbb{C}^d$. Then the spectral decomposition $\rho = \sum_{j=1}^{M} \lambda_j |\Psi_j\rangle\langle\Psi_j|$ is such that the eigenvalues $\lambda_1, \ldots, \lambda_M$ form a probability

distribution and the eigenvectors $|\Psi_1\rangle, \ldots, |\Psi_M\rangle$ form an orthogonal basis of $\mathbb{C}^M$. The von Neumann entropy $S(\rho)$ of ρ is defined by the equation

$$S(\rho) = -\sum_{j=1}^{M} \lambda_j \log_2 \lambda_j \,.$$

The von Neumann entropy takes its minimal value 0 on pure states, i. e., for $\rho = |\Psi\rangle\langle\Psi|$, and its maximal value $\log_2 M$ for the maximally mixed state $\rho = \mathbf{1}/M$. Let $U_1, \ldots, U_N \in \mathbb{C}^{M\times M}$ be arbitrary unitary matrices, $p_1, \ldots, p_N$ a probability distribution, and $|\Psi\rangle$ a state of $\mathbb{C}^M$. We have the following inequality (see [30], p. 518)

$$S\left(\sum_{j=1}^{N} p_j U_j^\dagger |\Psi\rangle\langle\Psi| U_j\right) \leq H(p_1, \ldots, p_N) \leq \log_2 N \,. \tag{22}$$

The following theorem shows that if a regular decoupling scheme can switch off an arbitrary t-local Hamiltonian, then the tensor products of the operations performed on an arbitrary t-tuple of qudits must form a unitary error basis for this subsystem.

Theorem 14 (Equivalence of decoupling schemes and OAs)
Let D be a regular decoupling scheme that uses elements of a unitary error basis $\mathcal{E} := \{U_1, \ldots, U_{d^2}\}$ as control operations, acts on n qudits and consists of N time-slots. Denote by $U_{m_{1j}} \otimes U_{m_{2j}} \otimes \cdots \otimes U_{m_{nj}}$ the local operation that is performed on the qudits in time-slot $j = 1, \ldots, N$, i. e., the indices $m_{kj} \in \{1, \ldots, d^2\}$ determine which elements of $\mathcal{E}$ are applied to the qudits in the time-slots. The scheme D can be used to decouple any t-body Hamiltonian if and only if the matrix $M = (m_{kj})$ where $k = 1, \ldots, n$ and $j = 1, \ldots, N$ is an orthogonal array $OA(N, n, d^2, t)$ of strength t.

Proof. Consider a fixed t-tuple $(k_1, \ldots, k_t)$ with different entries from $\{1, \ldots, n\}$. Let

$$H_{k_1\ldots,k_t} := \sum_{s=1}^{t} \sum_{(\ell_1,\ldots,\ell_s)} \sum_{\alpha_1\ldots\alpha_s} J_{(\ell_1,\ldots,\ell_s);\alpha_1\ldots\alpha_s} \sigma_{\alpha_1}^{(\ell_1)} \sigma_{\alpha_2}^{(\ell_2)} \cdots \sigma_{\alpha_s}^{(\ell_s)} \,,$$

where $(\ell_1, \ldots, \ell_s)$ runs over s-tuples with different entries from $\{k_1, \ldots, k_t\}$. We say that the operator $H_{k_1,\ldots,k_t}$ is the restriction of the t-body Hamiltonian H to the qudits $k_1, \ldots, k_t$. We denote by $\hat{H}_{k_1,\ldots,k_t}$ the corresponding operator acting on $(\mathbb{C}^d)^{\otimes t}$.

Note that for every traceless Hermitian operator X acting on $(\mathbb{C}^d)^{\otimes t}$ there is a t-local Hamiltonian H such that its restriction $H_{k_1,\ldots,k_t}$ to the qudits $k_1, \ldots, k_t$ is given by the embedding $X^{(k_1,\ldots,k_t)}$ of X to $(\mathbb{C}^d)^{\otimes n}$ according to the tuple $(k_1, \ldots, k_t)$. Let T_D be the operator

$$T_D : H \mapsto \sum_{i=1}^{N} p_j (U_{m_{1j}} \otimes U_{m_{2j}} \otimes \cdots \otimes U_{m_{nj}})^\dagger H (U_{m_{1j}} \otimes U_{m_{2j}} \otimes \cdots \otimes U_{m_{nj}})$$

Define the weight $w_{i_1,\ldots,i_t}$ of each tuple $(i_1,\ldots,i_t) \in \{1,\ldots,d^2\}^t$ to the sum of all p_j's with $(m_{k_1,j},\ldots,m_{k_s,j}) = (i_1,\ldots,i_t)$. Now suppose that $T_D(H) = 0$ for all t-body Hamiltonians. Consequently, we have that $T_D(H_{k_1,\ldots,k_t}) = 0$ for all restrictions to t-tuples. But this implies that the weights for all $\mathcal{A}^t$ must be equal. This is seen as follows: the equality

$$\begin{aligned} T_D(H_{k_1\ldots k_t}) &= \left[\sum_{i_1,\ldots,i_t=1}^{d^2} w_{i_1\ldots i_t}(U_{i_1}\otimes\cdots\otimes U_{i_t})^\dagger\,\hat{H}_{k_1\ldots k_t}\,(U_{i_1}\otimes\cdots\otimes U_{i_t})\right]^{(k_1,\ldots,k_t)} \\ &= 0^{(k_1,\ldots,k_t)} = 0 \end{aligned}$$

shows that the operation defined by the sum above is a unitary depolarizer for $(\mathbb{C}^d)^{\otimes t}$, i. e.,

$$\sum_{i_1,\ldots,i_t=1}^{d^2} w_{i_1\ldots i_t}(U_{i_1}\otimes\cdots\otimes U_{i_t})^\dagger\,X(U_{i_1}\otimes\cdots\otimes U_{i_t}) = \frac{tr(X)}{d^t}\mathbf{1}$$

for all operators X acting on $(\mathbb{C}^d)^{\otimes t}$. Now, we show that the weights must be all equal. Let $|\Psi_1\rangle,\ldots,|\Psi_{d^t}\rangle$ be an orthonormal basis of $(\mathbb{C}^d)^{\otimes t}$. We define a special state in the bipartite system $(\mathbb{C}^d)^{\otimes t}\otimes(\mathbb{C}^d)^{\otimes t}$ together with its corresponding density operator

$$|\Psi\rangle = \frac{1}{\sqrt{d^t}}\sum_{r=1}^{d^t}|\Psi_r\rangle\otimes|\Psi_r\rangle\,,\quad \rho = \frac{1}{d^t}\sum_{r,s=1}^{d^t}|\Psi_r\rangle\langle\Psi_s|\otimes|\Psi_r\rangle\langle\Psi_s|\,.$$

We use the fact that T_D is a unitary depolarizer to show that show that all weights are equal

$$\begin{aligned} \mathbf{1}\otimes T_D(\rho) &= \sum_{i_1\ldots i_t\in\mathcal{A}^t} w_{i_1\ldots i_t}(\mathbf{1}_{d^t}\otimes U_{i_1}\otimes\cdots\otimes U_{i_t})^\dagger\rho\,(\mathbf{1}_{d^t}\otimes U_{i_1}\otimes\cdots\otimes U_{i_t}) \\ &= \frac{1}{d^t}\sum_{r,s=1}^{d^t}|\Psi_r\rangle\langle\Psi_s|\otimes \\ &\qquad \sum_{i_1\ldots i_t\in\mathcal{A}^t} w_{i_1\ldots i_t}(U_{i_1}\otimes\cdots U_{i_t})^\dagger\,|\Psi_r\rangle\langle\Psi_s|\,(U_{i_1}\otimes\cdots U_{i_t}) \\ &= \frac{1}{d}\sum_{r=1}^{d}|\Psi_r\rangle\langle\Psi_r|\otimes\mathbf{1}_{d^t}/d^t \\ &= \mathbf{1}_{d^t}/d^t\otimes\mathbf{1}_{d^t}/d^t = \mathbf{1}_{d^{2t}}/d^{2t}\,. \end{aligned}$$

It follows from the above equation that we need at least d^{2t} different unitaries are necessary since the rank of each pure state $(\mathbf{1}\otimes U^\dagger)|\Psi\rangle\langle\Psi|(\mathbf{1}\otimes U)$ is one and since they have to sum up to d^{2t} (the rank of the maximally mixed state).

Since we use exactly d^{2t} different unitaries (tensor products of elements of the unitary error basis $\mathcal{E}$) as control operations all weights $w_{i_1\ldots i_t}$ must be equal due to the inequality (22). Now together with the fact that for regular schemes all time-slots have equal

length we conclude that $U_{i_1} \otimes \cdots \otimes U_{i_t}$ must appear with the same multiplicity. Therefore, by considering all t-tuples $k_1, \ldots, k_t$ of t qudits we see that the decoupling D scheme must correspond to an orthogonal array $OA(N, n)$ with d^2 levels and strength t. □

6 Conclusions

We have shown the equivalence between two constructions for decoupling schemes and selective coupling schemes in networks of qubits. One construction is based on triples of sign matrices which are closed under taking entry-wise products, while the other construction is based on orthogonal arrays of strength two over an alphabet of size four.

The construction using orthogonal arrays can be generalized to systems where the nodes have higher dimensions. Also the case where the system Hamiltonian has higher couplings can be dealt with by using orthogonal arrays: the coupling order directly translates into the strength of the orthogonal array. A special case arises when the local pulses which are applied in each time-slot are actually elements of a nice error basis for an abelian group. We have shown that in case of equidistant interval lengths (after refinement) this leads to a class of schemes which are equivalent to orthogonal arrays. In addition we have presented a construction of schemes for decoupling and selective coupling which can be constructed by using Hamming codes.

Moreover, we have shown that the construction of this particular class of decoupling and coupling schemes can be reduced to questions about the existence of these combinatorial arrays. While several constructions for orthogonal arrays are known, there still remain some open problems such as the case where the dimensions of the nodes could be different. Another important problem is to devise schemes for a situation where the given Hamiltonian is of a particular form, i. e., where not all interactions are present or can be assumed to be very weak for a large number of pairs. In this case a combination of graph theoretical techniques and the methods described in this paper can be developed.

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A Appendix: Characters of abelian groups

In this appendix we collect some basic facts of the representation theory of finite groups which are needed in the paper. Recall that $\mathrm{GL}(n,\mathbb{C})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$. Let G be a finite group. A *representation* of G over $\mathbb{C}$ (see also [31, 32]) is a homomorphism ρ from G to $\mathrm{GL}(n,\mathbb{C})$, for some $n \in \mathbb{N}$. The *degree* of ρ is given by n. The representation ρ is called *irreducible* if there are no invariant subspaces under the action of the matrices $\{\rho(g)\}_{g\in G}$ apart from the trivial subspaces $\{0\}$ and $\mathbb{C}^n$. With each $n \times n$ matrix $\rho(g)$ we associate the complex number given by the trace of $\rho(g)$, and call this number $\chi(g)$. The function $\chi : G \to \mathbb{C}$ is called the character of the representation ρ. A character is called irreducible if the corresponding representation is irreducible. We need the following theorem on abelian groups [32, Chap. V. §6].

Theorem 15 (Characters of abelian groups) *Let G be a finite abelian group of order $|G|$. Then every irreducible representation ρ of G has degree 1, i. e., we have that $\rho : G \to \mathbb{C}^\times$ is a homomorphism which maps G to scalars. Furthermore, the number of different irreducible representations (irreducible characters) of G is given by $|G|$ and the characters form a group $\hat{G} = \mathrm{Hom}(G,\mathbb{C}^\times)$ under pointwise multiplication. Hence, we have that*

$$\chi\tilde{\chi}(h) = \chi(h)\tilde{\chi}(h)$$

for all irreducible characters $\chi, \tilde{\chi}$ and $h \in G$. Moreover, the character group $\hat{G}$ is isomorphic to G. Thus, we can label the characters of G by the elements of G using an isomorphism which maps $h \mapsto \chi_h$ for all $h \in G$.

B Appendix: Proof of Lemmas 5 and 11

Lemma 5: *Let $\mathcal{E} := \{U_g \mid g \in G\}$ be a nice error basis with an abelian index group G. Then the matrix $\mathcal{X} = (\chi(g,h))_{g,h\in G}$ is the character table of the group G.*

Proof. Let α be the factor system corresponding to the nice error basis $\mathcal{E}$ with abelian index group G. We prove that $\mathcal{X}$ is a character table by showing that the rows of $\mathcal{X}$ form a group under pointwise multiplication that is isomorphic to G (see Theorem 15 in Appendix). We first show that

$$\chi(g,h) = \frac{\alpha(h,g)}{\alpha(g,h)}\,. \tag{23}$$

We have that

$$U_gU_h = \alpha(g,h)U_{gh}, \tag{24}$$
$$U_hU_g = \alpha(h,g)U_{hg} = \alpha(h,g)U_{gh}\,. \tag{25}$$

By multiplying eq. (25) by $U_g^\dagger$ from the left and using eq. (24) we obtain

$$U_g^\dagger U_h U_g = \alpha(h,g)U_g^\dagger U_{gh}$$

$$
\begin{aligned}
&= \frac{\alpha(h,g)}{\alpha(g,h)} U_g^\dagger U_g U_h \\
&= \frac{\alpha(h,g)}{\alpha(g,h)} U_h \,.
\end{aligned}
$$

We now prove that the rows of $\mathcal{X}$ form a group under pointwise multiplication that is isomorphic to G. Let $g, \tilde{g}$ be arbitrary elements of G. Note that we have $\overline{\alpha(\tilde{g}^{-1}, g)}\alpha(\tilde{g}^{-1}, g) = 1$ (otherwise the matrix $U_{\tilde{g}^{-1}} U_g = \alpha(\tilde{g}^{-1}, g) U_{\tilde{g}^{-1}g}$ would not be unitary). The group property is verified by

$$
\begin{aligned}
\chi(g,h)\chi(\tilde{g}^{-1},h)U_h &= U_g^\dagger U_{\tilde{g}^{-1}}^\dagger U_h U_{\tilde{g}^{-1}} U_g \\
&= \overline{\alpha(\tilde{g}^{-1},g)}\alpha(\tilde{g}^{-1},g)\, U_{g\tilde{g}^{-1}}^\dagger U_h U_{g\tilde{g}^{-1}} \\
&= U_{g\tilde{g}^{-1}}^\dagger U_h U_{g\tilde{g}^{-1}} \\
&= \chi(g\tilde{g}^{-1},h)U_h
\end{aligned}
$$

for all $h \in G$.

The rows of $\mathcal{X}$ form a group that is isomorphic to G (and not only to a proper subgroup of G) since there is a bijection between the rows of $\mathcal{X}$ and the elements of G. This is seen as follows. Assume that there are $g \neq \tilde{g}$ such that $\chi(g,h) = \chi(\tilde{g},h)$ for all $h \in G$. This is equivalent to $U_g^\dagger U_h U_g = U_{\tilde{g}}^\dagger U_h U_{\tilde{g}}$. Set $U = U_{\tilde{g}} U_g^\dagger$. Then we have $UM = MU$ for all $M \in \mathbb{C}^{d\times d}$ since the matrices U_h form a basis of $\mathbb{C}^{d\times d}$. Therefore U must be a multiple of the identity matrix. Due to the properties of a nice error basis this is only possible for $g = \tilde{g}$. This proves that there is a bijection between the group elements of G and the rows of $\mathcal{X}$. □

Lemma 11: *Let G be an abelian group of order $|G|$. Denote by $\chi_1, \chi_2, \ldots, \chi_{|G|}$ all irreducible characters of G, where χ_1 is the trivial character (i. e. $\chi_1(h) = 1$ for all $h \in G$). Let v be an arbitrary element of the group ring $\mathbb{C}[G]$, i. e., v is a formal sum of (weighted) group elements*

$$
v := \sum_{g\in G} \mu_g g \,, \quad \mu_g \in \mathbb{C} \,. \tag{26}
$$

If $\chi_i(v) = 0$ for all $i = 2, \ldots, |G|$ then we have $v = \frac{\mu}{|G|}\sum_{g\in G} g$, where $\mu := \chi_1(v) = \sum_{g\in G}\mu_g$.

Proof. Let $G := \{g_1, \ldots, g_{|G|}\}$ be an arbitrary ordering of the group elements, where g_1 is the identity element of G. Denote by $\mathcal{X}$ the (normalized) character table of G, i. e.,

$$
\mathcal{X}_{ij} := |G|^{-1/2}\, \chi_i(g_j) \tag{27}
$$

for $i, j = 1, \ldots, |G|$. Recall that the (normalized) character table $\mathcal{X}$ is a unitary matrix and has the following form [32, 31]

$$
\mathcal{X} = \frac{1}{|G|^{1/2}} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & * & \\ 1 & & \end{pmatrix} . \tag{28}
$$

The conditions given in the lemma can now be expressed as

$$|G|^{1/2}\mathcal{X}\begin{pmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_{|G|}\end{pmatrix}=\begin{pmatrix}\mu\\ 0\\ \vdots\\ 0\end{pmatrix}\,.$$

Multiplying by the inverse $\mathcal{X}^{-1}$ we obtain

$$(\mu_1,\mu_2,\ldots,\mu_{|G|})^T=\frac{\mu}{|G|}(1,1,\ldots,1)^T\,.$$

due to the special form in eq. (28). This show that all coefficients μ_g in eq. (20) are equal to $\mu/|G|$. □