Quantum algorithms for highly non-linear Boolean functions

Martin Rötteler
NEC Laboratories America
4 Independence Way, Suite 200
Princeton, NJ 08540, U.S.A.
mroetteler@nec-labs.com

November 24, 2009

Abstract

Attempts to separate the power of classical and quantum models of computation have a long history. The ultimate goal is to find exponential separations for computational problems. However, such separations do not come a dime a dozen: while there were some early successes in the form of hidden subgroup problems for abelian groups–which generalize Shor's factoring algorithm perhaps most faithfully–only for a handful of non-abelian groups efficient quantum algorithms were found. Recently, problems have gotten increased attention that seek to identify hidden sub-structures of other combinatorial and algebraic objects besides groups. In this paper we provide new examples for exponential separations by considering hidden shift problems that are defined for several classes of highly non-linear Boolean functions. These so-called bent functions arise in cryptography, where their property of having perfectly flat Fourier spectra on the Boolean hypercube gives them resilience against certain types of attack. We present new quantum algorithms that solve the hidden shift problems for several well-known classes of bent functions in polynomial time and with a constant number of queries, while the classical query complexity is shown to be exponential. Our approach uses a technique that exploits the duality between bent functions and their Fourier transforms.

1 Introduction

A salient feature of quantum computers is that they allow to solve certain problems much more efficiently than any classical machine. The ultimate goal of quantum computing is to find problems for which an exponential separations between quantum and classical models of computation can be shown in terms of the required resources such as time, space, communication, or queries. It turns out that the question about a provably exponential advantage of a quantum computer over classical computers is a challenging one and examples showing a separation are not easy to come by. Currently, only few (promise) problems giving an exponential separation between quantum and classical computing are known. A common feature they share is that, simply put, they all ask to extract hidden features of certain algebraic structures. Examples for this are hidden shift problems [vDHI03], hidden non-linear structures [CSV07], and hidden subgroup problems (HSPs). The latter class of hidden subgroup problems was studied quite extensively over the past decade. There are some successes such as the efficient solution of the HSP for any abelian group [Sho97, Kit97], including factoring and discrete log as well as Pell's equation [Hal02], and efficient solutions for some non-abelian groups [FIM^{+}03, BCvD05]. However, meanwhile some limitations of the known approaches to this problem are known [HMR$^+$06] and presently it is unclear whether the HSP can lend itself to a solution to other interesting problems such as the graph isomorphism problem.

Most of these methods invoke Fourier analysis over a finite group G. In some sense the Fourier transform is good at capturing some non-trivial global properties of a function f which at the same time are hard to figure out for the classical computer which can probe the function only locally at polynomially many places. For many groups G the quantum computer has the unique ability to compute a Fourier transform for G very efficiently, i. e., in time $\log^{O(1)} n$, where n is the input size. Even though the access to the Fourier spectrum is somewhat limited, namely via sampling, it nevertheless has been shown that this limited access can be quite powerful. Historically, the first promise problems which tried to leverage this power were defined for certain classes of Boolean functions: the Deutsch-Jozsa problem [DJ92] is to decide whether a Boolean function $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ that is promised to be either constant or a balanced function is actually constant or balanced. In the Fourier picture this asks to distinguish between functions that have all their spectrum supported on the 0 frequency and functions which have no 0 frequency component at all. It therefore comes as no surprise that by sampling from the Fourier spectrum the problem can be solved. Furthermore, it can be shown that any deterministic classical algorithm must make an exponential number of queries. However, this problem can be solved on a bounded error polynomial time classical machine. Hence other, more challenging, problems were sought which asked for more sophisticated features of the function f and were still amenable to Fourier sampling. One such problem is to identify $r \in \mathbb{Z}_2^n$ from black box access to a linear Boolean function $f(x) = rx$, where $x \in \mathbb{Z}_2^n$. Again, in the Fourier domain the picture looks very simple as each f corresponds to a perfect delta peak localized at frequency r, leading to an exact quantum algorithm which identifies r using a single query. Classically, it can be shown that $\Theta(n)$ queries are necessary and sufficient to identify r with bounded error. Based on the observation that a quantum computer can even handle the case well in which access to x is not immediate but rather through solving another problem of a smaller size, Bernstein and Vazirani [BV97] defined the recursive Fourier sampling (RFS) problem by organizing many instances of learning a hidden linear function in a tree-like fashion. By choosing the height of this tree to be $\log n$ they showed a separation between quantum computers, which can solve the problem in n queries, and classical computers which require $n^{\log n}$ queries. Soon after this, more algorithms were found that used the power of Fourier sampling over an abelian group, namely Simon's algorithm [Sim94] for certain functions $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{n-1}$, and Shor's algorithms [Sho97], where f was defined on cyclic groups and products thereof, eventually leading to the HSP.

The idea to achieve speedups from Boolean functions themselves has obtained significantly less attention. Recently, Hallgren and Harrow [HH08] revisited the RFS problem and showed that other unitary matrices can serve the role of the Fourier transform in the definition of RFS problems. They have obtained superpolynomial speedups over classical computing for a wide class of Boolean functions and unitary matrices, including random unitary matrices. Together with lower bound results [Aar03] this gives a reasonably good understanding of the power and limitations of the RFS problem. In another important development, it was shown that the ability to efficiently perform Fourier transforms on a quantum computer can also be used to efficiently perform correlations between certain functions. In the so-called hidden shift problem defined by van Dam, Hallgren, and Ip [vDHI03] this was used in the context of computing a correlation between a black box implementation of $f(x) = \left(\frac{x+s}{p}\right)$, where $\left(\frac{x}{p}\right)$ denotes the Legendre symbol and $s \in \mathbb{Z}_p$ is a fixed element, and the Legendre symbol itself. The main idea behind this is that the Fourier transform of a shifted function picks up a linear phase which depends on the shift. Since a correlation corresponds to point-wise multiplication of the Fourier transforms and since the Legendre symbol is its own Fourier transform, the correlation can be performed by computing the Legendre symbol into the phase, leading to an efficient algorithm that needs only a constant number of queries. The classical query complexity of this problem is polynomial in $\log p$.

Our results. Our main contribution is a generalization of the hidden shift problem for a class of Boolean functions known as bent functions [Rot76]. Bent functions are those Boolean functions for which the Hamming distance to the set of all linear Boolean functions is maximum (based on comparing their truth tables). For this reason bent functions are also called maximum non-linear functions.[1] A direct consequence of this is that the Fourier transform of a bent function f is perfectly flat, i. e., in absolute value all Fourier coefficients, which are defined with respect to the real valued function $x \mapsto (-1)^{f(x)}$, are equal and as small as possible. This feature of having a flat Fourier spectrum is desirable for cryptographic purposes because, roughly speaking, such a function is maximally resistant against attacks that seek to exploit a dependence of the outputs on some linear subspace of the inputs. It turns out that bent functions exist if and only if the number of variables is even and that there are many of them: asymptotically, the number of bent functions in n variables is at least $\Omega\left(\left(\frac{2^{n/2+1}}{e}\right)^{2^{n/2}}\sqrt{2\pi 2^{n/2}}\right)$, see for instance [CG06]. What is more, several explicit constructions of infinite families of bent functions are known and they are related to so-called difference sets which are objects studied in combinatorics. Since the Fourier transform of f is flat and the Boolean Fourier transform is real, it follows that (up to normalization) the Fourier spectrum takes only values ± 1, i. e., it again is described by a Boolean function, called the *dual* bent function and denoted by $\widetilde{f}$. Arguably, the most prominent example for a bent function is the inner product function $ip_n(x_1, \ldots, x_n) = \sum_{i=1}^{n/2} x_{2i-1}x_{2i}$ written in short as $ip_n(x,y) = xy^t$. This function can be generalized to $f(x,y) = x\pi(y)^t + g(y)$, where π is an arbitrary permutation of strings of length $n/2$ and $g : \mathbb{Z}_2^{n/2} \to \mathbb{Z}_2$ is an arbitrary function. This leads to the class of so-called Maiorana-McFarland bent functions. The dual bent function is then given by the Boolean function $\widetilde{f}(x,y) = \pi^{-1}(x)y^t + g(\pi^{-1}(x))$.

We define the hidden shift problem for a fixed bent function f as follows: an oracle $\mathcal{O}$ provides us with access to f and g, where g is promised to be a shifted version of f with respect to some unknown shift s. Using oracles of this kind, we show an exponential separation of the quantum and classical query complexity of the hidden shift problem, the former being at most linear, the latter being exponential. Furthermore, we also consider a variation of the problem where an oracle $\widetilde{\mathcal{O}}$ in addition provides oracle access to the dual bent function $\widetilde{f}$. We show that s can be extracted from $\widetilde{\mathcal{O}}$ by a quantum algorithm using one query to f and one query to $\widetilde{f}$. We present two other classes of bent functions, namely the partial spread class defined by Dillon [Dil75] and a class defined by Dobbertin [Dob95], which uses properties of certain Kloosterman sums over finite fields to show the bentness of the functions.

What is the significance of our result? In short, we provide new examples for exponential separations between quantum and classical computing. The class of problems studied in this paper yields a large new set of problems for exponential separations in query complexity with respect to oracles. A feature of the quantum algorithms presented here are their simplicity in that besides classical computation of function values the only quantum operation required are the Fourier transform over the groups $\mathbb{Z}_2^n$.

How does this relate to other separations? While exponential separations in query complexity were known before, for instance for abelian hidden subgroup problems, the hidden shift problems for bent functions are the first problems for which such a separation can be shown from *Boolean* functions. In the case of abelian HSP for order 2 subgroups of $\mathbb{Z}_2^n$, it is possible to assume that the functions hiding the hidden subgroup take the form $f(x) = \pi(Ax)$, where $A \in \mathbb{F}_2^{(n-1)\times n}$ is a matrix of rank $n-1$, and π is a permu-

[1] Note that high nonlinearity of a function refers to the spectral characterization, i. e., the Hamming weight of the highest nonzero frequency component is high. It does not imply that $f(x) = \sum_{\nu \in \mathbb{Z}_2^n} \alpha_\nu x^\nu$, when written as a multivariate polynomial over $\mathbb{F}_2$, has a high (algebraic) degree, defined as the maximum degree of any monomial x^ν. Indeed, there are many examples of highly nonlinear functions whose algebraic degree is 2.

tation of strings of length $n-1$. The goal is to find a vector $s \in \mathbb{F}_2^n$ in the kernel of A. Note that these functions are not Boolean functions but rather functions from $\mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{n-1}$. To the best of our knowledge the best separations that were obtainable so far from Boolean functions were the superpolynomial separations shown in [HH08]. Those were obtained by generalizing the ideas of recursive Fourier sampling from parity functions to more general classes of Boolean functions.

Related work. The techniques used in this paper are related to the techniques used in [vDHI03], in particular the method of using the Fourier transform thrice in order to correlate a shifted function with a given reference function, thereby solving a deconvolution problem. We see the main difference in the richness of the class of Boolean functions for which the method can be applied and the query lower bound.

It was observed in [FIM+03, Kup05] that the hidden shift problem for *injective* functions $f, g : G \rightarrow S$ from an abelian G to a set S is equivalent to hidden subgroup problem over $G \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ on G is given by the inverse. There are several other papers that deal with the injective hidden shift problem over abelian and non-abelian groups [CvD07, CW07, MRRS07]. In contrast, the functions studied here are defined on the abelian group $\mathbb{Z}_2^n$ and very far from being injective. As we show it will be nevertheless possible to define a related hidden subgroup problem over an elementary abelian group, however, for this we have to consider “quantum functions” to encode the period.

Perhaps most closely related to our scenario is the work by Russell and Shparlinski [RS04] who considered shift problems for the case of $\chi(f(x))$, where f is a polynomial on a finite group G and χ a character of G, a general setup that includes our scenario. The two cases for which algorithms were given in [RS04] are the reconstruction of a monic, square-free polynomial $f \in \mathbb{F}_p[X]$, where χ is the quadratic character (Legendre symbol) over $\mathbb{F}_p$ and the reconstruction of a hidden shift over a finite group $\chi(sx)$, where χ is the character of a known irreducible representation of G. The technique used in [RS04] is a generalization of the technique of [vDHI03]. In the present paper we extend the class of functions for which the hidden shift problem can be solved to the case where f is a multivariate polynomial and G is the group $\mathbb{Z}_2^n$.

Related to the hidden shift problem is the problem of unknown shifts, i. e., problems in which we are given a supply of quantum states of the form $|D+s\rangle$, where s is random, and D has to be identified. Problems of this kind have been studied by Childs, Vazirani, and Schulman [CSV07], where D is a sphere of unknown radius, Decker, Draisma, and Wocjan [DDW08], where D is a graph of a function, and Montanaro [Mon09], where D is the set of points of a fixed Hamming-weight. The latter paper also considers the cases where D hides other Boolean functions such as juntas, a problem that was also studied in [AS07]. In contrast to all these problems in our case the set D is already known, but the shift s has to be identified.

We are only aware of relatively few occasions where bent functions have been used in theoretical computer science: they were used in the context of learning of intersections of halfspaces [KS07], where they gave rise to maximum possible number of slicings of edges of the hypercube. Also the recent counterexample for failure of the inverse Gowers conjecture in small characteristic [LMS08] uses a special bent function.

2 Fourier analysis of Boolean functions

We recall some basic facts about Fourier analysis of Boolean functions, see also the recent review article [dW08] for an introduction. Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ be a real valued function on the n-dimensional Boolean hypercube. The Fourier representation of f is defined as follows. First note that for any subset $S \subseteq [n] = \{1, \ldots, n\}$ we can define a character of $\mathbb{Z}_2^n$ via $\chi_S : x \mapsto (-1)^{Sx^t}$, where $x \in \mathbb{Z}_2^n$ (the transpose is necessary as we assume that all vectors are row vectors). The inner product of two functions on the hypercube is defined as $\langle f, g\rangle = \frac{1}{2^n}\sum_x f(x)g(x) = \mathbb{E}_x(fg)$. The χ_S are inequivalent characters of $\mathbb{Z}_2^n$, hence they obey the orthogonality relation $\mathbb{E}_x(\chi_S\chi_T) = \delta_{S,T}$. The Fourier transform of f is a function $\widehat{f} : \mathbb{Z}_2^n \rightarrow \mathbb{R}$ defined by

$$\widehat{f}(S) = \mathbb{E}_x(f\chi_S) = \frac{1}{2^n}\sum_{x\in\mathbb{Z}_2^n}\chi_S(x)f(x), \tag{1}$$

$\widehat{f}(S)$ is the Fourier coefficient of f at frequency S, the set of all Fourier coefficients is called the Fourier spectrum of f and we have the representation $f = \sum_S \widehat{f}(S)\chi_S$. Two useful facts about the Fourier transform of Boolean functions are Parseval's identity and the convolution property. Parseval's identity says that $\|f\|_2^2 = \sum_S \widehat{f}(S)^2$ which is a special case of $\langle f, g\rangle = \sum_S \widehat{f}(S)\widehat{g}(S)$. For two Boolean functions $f, g : \mathbb{Z}_2^n \to \mathbb{R}$ their convolution $(f * g)$ is the function defined as $(f * g)(x) = \frac{1}{2^n}\sum_{y\in\mathbb{Z}_2^n} f(x+y)g(y)$. A standard feature of the Fourier transform is that it maps the group operation to a point wise operation in the Fourier domain. Concretely, this means that $\widehat{f * g}(S) = \widehat{f}(S)\widehat{g}(S)$, i. e., convolution becomes point-wise multiplication and vice-versa.

In quantum notation the Fourier transform on the Boolean hypercube differs slightly in terms of the normalization and is given by the unitary matrix

$$H_{2^n} = \frac{1}{\sqrt{2^n}}\sum_{x,y\in\mathbb{Z}_2^n}(-1)^{xy^t}\left|x\right\rangle\left\langle y\right|.$$

This is sometimes called the Hadamard transform [NC00]. In this paper we will also use the Fourier spectrum defined with respect to the Hadamard transform which differs from (1) by a factor of $2^{-n/2}$. It is immediate from the definition of H_{2^n} that it can be written in terms of a tensor (Kronecker) product of the Hadamard matrix of size 2×2, namely $H_{2^n} = (H_2)^{\otimes n}$, a fact which makes this transform appealing to use on a quantum computer since it can be computed using $O(n)$ elementary operations. Also note that in the context of cryptography also the name Walsh-Hadamard transform for H_{2^n} is common.

Another note on a convention which applies when we consider $\mathbb{Z}_2$ valued functions $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$. Then we tacitly assume that the real valued function corresponding to f is actually $F : x \mapsto (-1)^{f(x)}$. The Fourier transform is then defined with respect to F, i. e., we obtain that

$$\widehat{F}(w) = \frac{1}{2^n}\sum_{x\in\mathbb{Z}_2^n}(-1)^{wx^t+f(x)}, \tag{2}$$

where we use $w \in \mathbb{Z}_2^n$ instead of $S \subseteq [n]$ to denote the frequencies. Other than this notational convention, the Fourier transform used in (2) for Boolean valued functions and the Fourier transform used in (1) for real valued functions are the same. In the paper we will sloppily identify $\widehat{f} = \widehat{F}$ and it will be clear from the context which definition has to be used.

3 Bent functions

Definition 1. *Let $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ be a Boolean function. We say that f is* bent *if the Fourier coefficients $\widehat{f}(w) = \frac{1}{2^n}\sum_{x\in\mathbb{Z}_2^n}(-1)^{wx^t+f(x)}$ satisfy $|\widehat{f}(w)| = 2^{-n/2}$ for all $w \in \mathbb{Z}_2^n$, i. e., if the spectrum of f is flat.*

Necessary for bent functions in n variables to exist is that n is even [Dil75, MS77]. If f is bent, then this implicitly defines another Boolean function via $2^{n/2}\widehat{f}(w) =: (-1)^{\widetilde{f}(w)}$. Then this function $\widetilde{f}$ is again a bent function and called the dual bent function of f. By taking the dual twice we obtain f back: $\widetilde{\widetilde{f}} = f$.

3.1 A first example: the inner product function

The most simple bent function is $f(x,y) := xy$ where $x,y \in \mathbb{Z}_2$. It is easy to verify that f defines a bent function. This can be generalized to $2n$ variables [MS77] and we obtain the inner product

$$ip_n(x_1,\ldots,x_n,y_1,\ldots,y_n) := \sum_{i=1}^{n} x_i y_i.$$

Again, it is easy to see that ip_n is bent. In Section 3.2 we will see that ip_n belongs to a much larger class of bent functions. There (in Lemma 4) we also establish that that $ip_n = \widetilde{ip_n}$ is its own dual bent function which also implies that the vector $[(-1)^{ip_n(x,y)}]_{x,y\in\mathbb{Z}_2^n}$ is an eigenvector of H_{2^n}. This should be compared to [vDHI03] where it was used that the Legendre symbol $\left(\frac{\cdot}{p}\right)$ gives rise to an eigenvector of the Fourier transform DFT_p over the cyclic group $\mathbb{Z}_p$. The shift problem for the inner product function is closely related to the Fourier sampling problem of finding a string a that is hidden by the function $f(a,x) = ax^t$ [BV97], and indeed the string a can be readily identified from the state $\frac{1}{\sqrt{2^n}}\sum_{x\in\mathbb{Z}_2^n}(-1)^{ax^t}\left|x\right\rangle$. In the hidden shift problem the problem is to identify (a,b) from $\frac{1}{2^n}\sum_{x,y\in\mathbb{Z}_2^n}(-1)^{ip_n(x+a,y+b)}\left|x,y\right\rangle$. This state is up to a global phase given by $\frac{1}{2^n}\sum_{x,y\in\mathbb{Z}_2^n}(-1)^{xy^t+xb^t+ya^t}\left|x,y\right\rangle$. By computing ip_n into the phase the latter can be mapped to $\frac{1}{2^n}\sum_{x,y\in\mathbb{Z}_2^n}(-1)^{xb^t+ya^t}\left|x,y\right\rangle$. From this state the string (a,b) can be extracted by applying to it a Boolean Fourier transform followed by measurement in the computational basis.

3.2 Bent function families

Many examples of bent functions are known and we briefly review some of these classes. Recall that any quadratic Boolean function f has the form $f(x_1,\ldots,x_n) = \sum_{i<j} q_{i,j}x_ix_j + \sum_i \ell_i x_i$ which can be written as $f(x) = xQx^t + Lx^t$, where $x = (x_1,\ldots,x_n) \in \mathbb{Z}_2^n$. Here, $Q \in \mathbb{F}_2^{n\times n}$ is an upper triangular matrix and $L \in \mathbb{F}_2^n$. Note that since we are working over the Boolean numbers, we can without loss of generality assume that the diagonal of Q is zero (otherwise, we can absorb the terms into L). It is useful to consider the associated symplectic matrix $B = (Q + Q^t)$ with zero diagonal which defines a symplectic form $\mathcal{B}(u,v) = uBv^t$. This form is non-degenerate if and only if $\mathrm{rank}(B) = n$. The coset of $f + R(n,1)$ of the first order Reed-Muller code is described by the rank of B. This follows from Dickson's theorem [MS77] which gives a complete classification of symplectic forms over $\mathbb{Z}_2$:

Theorem 1 (Dickson)**.** *Let $B \in \mathbb{Z}_2^{n\times n}$ be a symmetric matrix with zero diagonal (such matrices are also called symplectic matrices). Then there exists $R \in \mathrm{GL}(n,\mathbb{Z}_2)$ and $h \in [n/2]$ such that $RBR^t = D$, where D is the matrix $(\mathbf{1}_h \otimes \sigma_x) \oplus \mathbf{0}_{n-2h}$ considered as a matrix over $\mathbb{Z}_2$ (where σ_x is the permutation matrix corresponding to $(1,2)$). In particular, the rank of B is always even. Furthermore, under the base change given by R the function f becomes the quadratic form $ip_h(x_1,\ldots,x_{2h}) + L'(x_1,\ldots,x_n)$ where we used the inner product function ip_h and a linear function L'.*

Next, we give a characterization of the Fourier transform of an affine transform of a bent function.

Lemma 2 (Affine transforms)**.** *Let f be a bent function, let $A \in \mathrm{GL}(n,\mathbb{Z}_2)$ and $b \in \mathbb{Z}_2^n$, and define $g(x) := f(xA+b)$. Then also $g(x)$ is a bent function and $\widehat{g}(w) = (-1)^{-w(A^{-1})^t b}\widehat{f}(w(A^{-1})^t)$ for all $w \in \mathbb{Z}_2^n$.*

Proof. We compute $\widehat{g}(w)$ using the substitution $y = xA + b$ as follows:

$$\begin{aligned}
\widehat{g}(w) &= \frac{1}{2^n}\sum_x (-1)^{wx^t+f(xA+b)} \\
&= \frac{1}{2^n}\sum_y (-1)^{w\cdot(A^{-1})^t(y-b)^t+f(y)} \\
&= \frac{1}{2^n}(-1)^{-w(A^{-1})^t b}\sum_y (-1)^{w(A^{-1})^t y^t+f(y)} \\
&= (-1)^{-w(A^{-1})^t b}\widehat{f}(w(A^{-1})^t).
\end{aligned}$$

□

By combining Theorem 1 and Lemma 2 we arrive at the following corollary which characterizes the class of quadratic bent functions.

Corollary 3. *Let $f(x) = xQx^t + Lx^t$ be a quadratic Boolean function such that the associated symplectic matrix $B = (Q + Q^t)$ satisfies $\mathrm{rank}(B) = 2h = n$. Then f is a bent function. The dual of this bent function is again a quadratic bent function.*

A complete classification of all bent functions has only been achieved for $n = 2, 4$, and 6 variables. For larger number of variables some families are known, basically coming from ad hoc constructions. We present another one of the known families called **M** (Maiorana and McFarland). First, we remark there are also constructions for making new bent functions from known ones, the simplest one takes two bent functions f and g in n and m variables and outputs $(x, y) \mapsto f(x) \oplus g(y)$. The class **M** of Maiorana-McFarland bent functions consists of the functions $f(x, y) := x\pi(y)^t + g(y)$, where π is an arbitrary permutation of $\mathbb{Z}_2^n$ and g is an arbitrary Boolean function depending on y only. The following lemma characterizes the dual of a bent function in **M**.

Lemma 4. *Let $f(x, y) := x\pi(y)^t + g(y)$ be a Maiorana-McFarland bent function. Then the dual bent function of f is given by $\widetilde{f}(x, y) = \pi^{-1}(x)y^t + g(\pi^{-1}(x))$.*

Proof. Let $\widehat{f}(u, v)$ be the Fourier transform of f at $(u, v) \in \mathbb{Z}_2^{2n}$. We obtain

$$\begin{aligned}
\widehat{f}(u,v) &= \frac{1}{2^{2n}}\sum_{x,y\in\mathbb{Z}_2^n} (-1)^{f(x,y)+(u,v)(x,y)^t} \\
&= \frac{1}{2^{2n}}\sum_{x,y\in\mathbb{Z}_2^n} (-1)^{x\pi(y)^t+g(y)+(u,v)(x,y)^t} \\
&= \frac{1}{2^{2n}}\sum_{y\in\mathbb{Z}_2^n} (-1)^{vy^t+g(y)}\left(\sum_{x\in\mathbb{Z}_2^n}(-1)^{(u+\pi(y))x^t}\right) \\
&= \frac{1}{2^n}\sum_{y\in\mathbb{Z}_2^n} (-1)^{vy^t+g(y)}\delta_{u,\pi(y)} \\
&= \frac{1}{2^n}(-1)^{v\pi^{-1}(u)^t+g(\pi^{-1}(u))}.
\end{aligned}$$

Hence the dual bent function is given by $\widetilde{f}(x, y) = \pi^{-1}(x)y^t + g(\pi^{-1}(x))$. □

Another class of bent functions called **PS** (partial spreads) was introduced by Dillon [Dil75] and provides examples of bent functions outside of **M**.

Theorem 5. *[Dil75] Let $U_1, \ldots, U_{2^{n/2-1}}$ be $n/2$-dimensional subspaces of $\mathbb{Z}_2^n$ such that $U_i \cap U_j = \{0\}$ holds for all $i \neq j$. Let χ_i be the characteristic function of U_i. Then $f := \sum_{i=1}^{2^{n/2-1}} \chi_i$ is a bent function.*

A collection of sets U_i as in Theorem 5 is called a *partial spread*. Explicitly, the U_i can be chosen as $U_i = \{(x, a_i x) : x \in \mathbb{F}_{2^{n/2}}\}$ where $a_i \in \mathbb{F}_{2^{n/2}}^\times$ satisfies $g(a_i) = 1$ for a fixed balanced function g. Here we have identified $\mathbb{Z}_2^n$ with the finite field $\mathbb{F}_{2^n}$ by choosing a polynomial basis. This provides an explicit construction for bent functions in $\mathbf{PS}$. A further class defined by Dobbertin has the property to include $\mathbf{M}$ and $\mathbf{PS}$ is defined as follows: first, identify $\mathbb{Z}_2^n$ with $\mathbb{F}_{2^{n/2}} \times \mathbb{F}_{2^{n/2}}$. Let g be a balanced Boolean function of $n/2$ variables, φ be a permutation of $\mathbb{F}_{2^{n/2}}$ and ψ be an arbitrary map from $\mathbb{F}_{2^{n/2}}$ to $\mathbb{F}_{2^{n/2}}$. Then

$$f(x,y) := \begin{cases} g\left(\frac{x+\psi(\varphi^{-1}(y))}{\varphi^{-1}(y)}\right) & : \text{if } y \neq 0, \\ 0 & : \text{if } y = 0 \end{cases}$$

is a bent function.

There are other constructions of bent functions by means of so-called trace monomials. For this connection, an understanding of certain Kloosterman sums turns out to be important. Recall that the Kloosterman sum in $\mathbb{F}_{2^n}$ is defined as $Kl(a) = \sum_{x \in \mathbb{F}_{2^n}^\times} (-1)^{\mathrm{tr}(x^{-1}+ax)}$, where $\mathbb{F}_{2^n}^\times$ denotes the non-zero elements of $\mathbb{F}_{2^n}$ and tr denotes the trace map from $\mathbb{F}_{2^n}$ to $\mathbb{Z}_2$. For $a \in \mathbb{F}_{2^n}$ let $f_a(x)$ be the Boolean function $f_a(x) = \mathrm{tr}(ax^{2^{n/2-1}})$. It is known that if a is contained in the subfield $\mathbb{F}_{2^{n/2}}$ and $Kl(a) = -1$, then f_a is a bent function [Dil75]. The existence of such an element a was conjectured in Dillon's paper and was proved in [LW90] (see also [HZ99]) where its existence was shown for all n, thereby showing existence bent functions in this class of trace monomials.

3.3 Other characterizations of bent functions

Finally, we note that there are many other characterizations of bent functions via other combinatorial objects, in particular difference sets. The connection is rather simple: we get that $D_f := \{x : f(x) = 1\}$ is a difference set in $\mathbb{Z}_2^n$, i. e., the set $\Delta D_f = \{d_1 - d_2 : d_1, d_2 \in D_f\}$ of differences covers each non-zero element of $\mathbb{Z}_2^n$ an equal number of times. We briefly highlight some other connections to combinatorial objects in the following:

Circulant Hadamard matrices. Bent functions give rise to Hadamard matrices of size $2^n \times 2^n$ in a very natural way as group circulants as follows. Let $A_f := ((-1)^{f(x+y)})_{x,y \in \mathbb{Z}_2^n}$, then f is bent if and only if A_f is a Hadamard matrix, i. e, $A_f A_f^\dagger = n\mathbf{1}_n$. Another way of saying this is that the shifted functions $x \mapsto (-1)^{f(x+s)}$ for $s \in \mathbb{Z}_2^n$ are orthogonal. Moreover, in the basis given by the columns of H_{2^n} the matrix A_f becomes diagonal, the diagonal entries being $\widetilde{f}(x)$.

Balanced derivatives. Besides the property of A_f being a Hadamard matrix another equivalent characterizations of f to be bent is that the function $\Delta_h(f) := f(x+h) + f(x)$ is a balanced Boolean function (i. e., f takes 0 and 1 equally often) for all non-zero h.

Reed-Muller codes. Bent functions can also be characterized in terms of the Reed-Muller codes [MS77]. Recall that the set of all truth tables (evaluations) of all polynomials over $\mathbb{Z}_2$ of degree up to r in n variables is called the Reed-Muller $R(n,r)$. Then bent functions correspond to functions which have the maximum possible distance to all linear functions, i. e., elements of $R(n,1)$. Quadratic bent functions in $R(n,2)$ are of particular interest. They correspond to symplectic forms of maximal rank and play a role, e. g., in the definition of the Kerdock codes.

Difference sets. Finally, we note that bent functions are equivalent to objects known as difference sets in combinatorics, namely difference sets for the elementary abelian groups $\mathbb{Z}_2^n$ [BJL99]. A difference set is defined as follows: Let G be a finite group of order $v = |G|$. A (v,k,λ)-difference set in G is a subset $D \subseteq G$ such that the following properties are satisfied: $|D| = k$ and the set $\Delta D = \{a - b : a, b \in D, a \neq b\}$ contains every element in G precisely λ times. Examples for difference sets are for instance the set $D = \{x^2 : x \in \mathbb{F}_q\}$ of all squares in a finite field. Here the group G is the additive group of $\mathbb{F}_q$, where $q \equiv 3 \pmod 4$ is a prime power. The parameters of this family of difference sets is given by $(q, \frac{q-1}{2}, \frac{q-3}{4})$. Bent functions on the other hand give rise to difference sets in the elementary abelian group $G = \mathbb{Z}_2^n$. The connection is as follows: $D_f := \{x : f(x) = 1\}$ is a difference set in $\mathbb{Z}_2^n$ if and only if f is a bent function, a result due to Dillon [Dil75]. In this fashion we obtain $(2^n, 2^{n-1} \pm 2^{(n-2)/2}, 2^{n-2} \pm 2^{(n-2)/2})$ difference sets in $\mathbb{Z}_2^n$, see also [BJL99].

4 Quantum algorithms for the shifted bent function problem

We introduce the hidden shift problem for Boolean functions. In general, the hidden shift problem is a quite natural source of problems for which a quantum computer might have an advantage over a classical computer. See [CvD08] for more background on hidden shifts and related problems.

Definition 2 (Hidden shift problem)**.** *Let $n \geq 1$ and let $\mathcal{O}_f$ be an oracle which gives access to two Boolean functions $f, g : \mathbb{Z}_2^n \to \mathbb{Z}_2$ such that the following conditions hold: (i) f, and g are bent functions, and (ii) there exist $s \in \mathbb{Z}_2^n$ such that $g(x) = f(x+s)$ for all $x \in \mathbb{Z}_2^n$. We then say that $\mathcal{O}_f$ hides an instance of a shifted bent function problem for the bent function f and the hidden shift $s \in \mathbb{Z}_2^n$. If in addition to f and g the oracle also provides access to the dual bent function $\widetilde{f}$, then we use the notation $\mathcal{O}_{f,\widetilde{f}}$ to indicate this potentially more powerful oracle.*

Theorem 6. *Let $\mathcal{O}_{f,\widetilde{f}}$ be an oracle that hides an instance of a shifted bent function problem for a function f and hidden shift s and provides access to the dual bent function $\widetilde{f}$. Then there exists a polynomial time quantum algorithm $\mathcal{A}_1$ that computes s with zero error and makes two quantum queries to $\mathcal{O}_{f,\widetilde{f}}$.*

Proof. Let $f : \mathbb{Z}_2^n \to \mathbb{Z}_2$ be the bent function. We have oracle access to the shifted function $g(x) = f(x+s)$ via the oracle, i. e., we can apply the map $|x\rangle |0\rangle \mapsto |x\rangle |f(x+s)\rangle$ where $s \in \mathbb{Z}_2^n$ is the unknown string. Recall that whenever we have a function implemented as $|x\rangle |0\rangle \mapsto |x\rangle |f(x)\rangle$, we can also compute f into the phase as $U_f : |x\rangle \mapsto (-1)^{f(x)} |x\rangle$ by applying f to a qubit initialized in $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. The hidden shift problem is solved by the following algorithm $\mathcal{A}_1$: (i) Prepare the initial state $|0\rangle$, (ii) apply the Fourier transform $H_2^{\otimes n}$ to prepare an equal superposition $\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} |x\rangle$ of all inputs, (iii) compute the shifted function g into the phase to get $\frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{Z}_2^n} (-1)^{f(x+s)} |x\rangle$, (iv) Apply $H_2^{\otimes n}$ to get $\sum_w (-1)^{sw^t} \widehat{f}(w) |w\rangle = \frac{1}{\sqrt{2^n}} \sum_w (-1)^{sw^t} (-1)^{\widetilde{f}(w)} |w\rangle$, (v) compute the function $|w\rangle \mapsto (-1)^{\widetilde{f}(w)} |w\rangle$ into the phase resulting in $\frac{1}{\sqrt{2^n}} \sum_w (-1)^{sw^t} |w\rangle$, where we have used the fact that f is a bent function, and (vi) finally apply another

Hadamard transform $H_2^{\otimes n}$ to get the state $|s\rangle$ and measure s. From this description it is clear that we needed one query to g and one query to $\widetilde{f}$ to solve the problem, that the algorithm is exact, and that the overall running time is given by $O(n)$ quantum operations. A quantum circuit implementing this algorithm is shown in Figure 1(a). $\square$

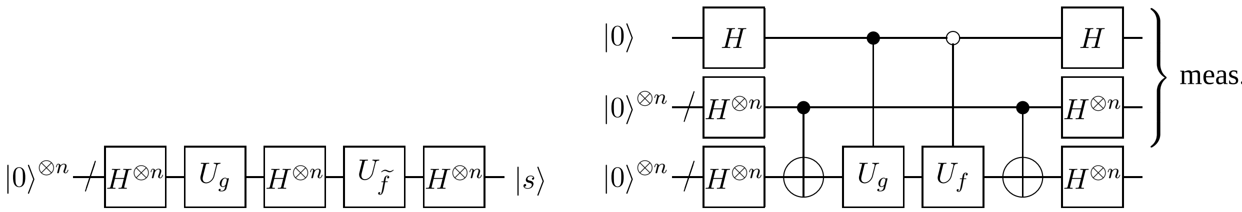
Next, we consider the situation where the oracle defines a hidden shift problem but does not provide access to the dual bent function. It turns out that in this case we can still extract the hidden shift with a polynomial time quantum algorithm, however the number of queries increases from constant to linear.

Theorem 7. *Let $\mathcal{O}_f$ be an oracle that hides an instance of a shifted bent function problem for a function f and hidden shift s. Then there exists a polynomial time quantum algorithm $\mathcal{A}_2$ that computes s with constant probability of success and makes $O(n)$ queries to $\mathcal{O}_f$.*

Proof. First, note that as in Theorem 6 we can assume that the oracle computes the functions $f, g : \mathbb{Z}_2^n \to \mathbb{Z}_2$ into the phase. Furthermore, we can assume that the oracle can be applied conditionally on a bit b, i. e., we can apply the map $\Lambda_1(U_f) : |b\rangle |x\rangle \mapsto |b\rangle |x\rangle$ if $b = 0$ and $|b\rangle |x\rangle \mapsto |b\rangle (-1)^{f(x)} |x\rangle$ if $b = 1$. Indeed, using a Fredkin gate $\textsc{Fred}$ (see [NC00]) which specified by $|b\rangle |x\rangle |y\rangle \mapsto |b\rangle |x\rangle |y\rangle$ if $b = 0$ and $|b\rangle |x\rangle |y\rangle \mapsto |b\rangle |y\rangle |x\rangle$ if $b = 1$, it is easy to implement $\Lambda_1(U_f)$ as follows: $(\Lambda_1(U_f) \otimes \mathbf{1}_{2^n}) |b\rangle |x\rangle |0\rangle = (\textsc{Fred} \circ (\mathbf{1}_2 \otimes U_f \otimes \mathbf{1}_{2^n}) \circ \textsc{Fred}) |b\rangle |x\rangle |0\rangle$, up to a global phase.

We prove the theorem by reducing to an abelian hidden subgroup problem in the group $\mathbb{Z}_2^{n+1}$. To do this, we use f and g to define "quantum functions", namely $F : x \mapsto \sum_{y \in \mathbb{Z}_2^n} (-1)^{f(x+y)} |y\rangle$ and $G : x \mapsto \sum_{y \in \mathbb{Z}_2^n} (-1)^{g(x+y)} |y\rangle$. Observe that due to the bentness of f and g, the two functions F and G are injective quantum functions, i. e., they are injective complex valued functions that with respect to some basis, which in general might be different from the computational basis, become classical injective functions. Indeed, this follows from the fact that all derivatives of a bent function are balanced, see Section 3.2. Now, a well known connection between the hidden shift problem for injective functions f, g over an abelian group A and a hidden subgroup problem can be used [Kup05, FIM$^+$03]. For this, the hidden subgroup problem is defined with respect to the semidirect product $A \rtimes \mathbb{Z}_2$ where the action is given by inversion in A. In our case we have $A \rtimes \mathbb{Z}_2 \cong Z_2^{n+1}$ since the inversion action is trivial over $\mathbb{Z}_2$. The hiding function for the HSP over Z_2^{n+1} is defined as $H(b, x) = F(x)$ if $b = 0$ and $H(b, x) = G(x)$ if $b = 1$. This defines a hidden subgroup $\{(0, 0), (1, s)\}$ of order 2, knowledge of which clearly implies that we know s. Once we have shown how to implement the hiding function H, the algorithm will therefore be the standard algorithm for the HSP: (i) Prepare the initial state $|0\rangle$, (ii) apply the Fourier transform $H_2^{\otimes n}$ to prepare an equal superposition $\frac{1}{\sqrt{2^{n+1}}} \sum_{b \in \mathbb{Z}_2, x \in \mathbb{Z}_2^n} |x\rangle$ of all inputs, (iii) compute the function into the second register to get $\frac{1}{\sqrt{2^{n+1}}} \sum_{b \in \mathbb{Z}_2, x \in \mathbb{Z}_2^n} |b, x\rangle |H(b, x)\rangle$, (iv) Apply $H_2^{\otimes n+1}$ to the first register, and (v) measure the first register. This leads to a measurement result $a \in \mathbb{Z}_2^{n+1}$ that satisfies $(1, s)a^t = 0$. Repeating steps (i)-(v) a total number of $O(n)$ times, we get a constant probability to uniquely characterize s from the measurement data. Hence, the algorithm needs $O(n)$ queries to f and g to solve the problem and the overall running time is given by $O(n^2)$ quantum operations. The function $H(b, x)$ can be implemented in a straightforward way using Hadamard transforms, controlled NOT operations [NC00], and the controlled oracle calls $\Lambda_1(U_f)$ mentioned above. A quantum circuit implementing one iteration of this algorithm is shown in Figure 1(b). $\square$

It is perhaps interesting to note that the "probabilistic method" of directly implementing $\widetilde{f}$ via sampling of f at a polynomial number of inputs and using the Chernoff bound is not sufficient for our purposes (see e. g., [Man94] for the argument that $\sum_{i \in I} \chi_S(x_i) f(x_i)$ is exponentially close to $\widetilde{f}$ for all S for a sample set



(a) Quantum algorithm $\mathcal{A}_1$

(b) Quantum algorithm $\mathcal{A}_2$

Figure 1: Quantum algorithms for the hidden shift problem for bent functions. The quantum circuit in (a) implements algorithm $\mathcal{A}_1$. This algorithm can be used if access to the shifted function $g(x) = f(x+s)$ as well as access to the dual bent function $\widetilde{f}$ is given. The algorithm uses one query to g and one query to $\widetilde{f}$ and is zero-error, i. e., it always returns the hidden shift s. The quantum circuit in (b) implements algorithm $\mathcal{A}_2$. This algorithm uses access to f and g only and can be applied if access to $\widetilde{f}$ is not available. The shown circuit has to be applied $O(n)$ times, after which the data acquired by measuring the upper $n+1$ qubits characterizes the hidden shift s with constant probability of success.

I of polynomial size). The issue is that for bent functions we would have to distinguish exponentially small Fourier coefficients $\pm 1/\sqrt{2^n}$. We conjecture that in the worst case it takes an exponential number of queries to f in order to implement one query to $\widetilde{f}$, but have no proof for this.

Finally, we state the two results that provide new query complexity separations between quantum and classical algorithms. Our main tool is the Maiorana-McFarland class of bent functions which turns out to be rich enough to prove the two results. First, we show that the classical query complexity for the hidden shift problem over this class of bent functions is of order $\Theta(n)$, while it can be solved with 2 quantum queries.

Theorem 8. *Let $\mathcal{O}_{f,\widetilde{f}}$ be an oracle that hides a hidden shift s for an instance $(f, g, \widetilde{f})$ of a hidden shift problem for a bent function f from Maiorana-McFarland class. Then classically $\Theta(n)$ queries are necessary and sufficient to identify the hidden shift s. Further, there exists a recursively defined oracle $\mathcal{O}_{rec}$ which makes calls to $\mathcal{O}_{f,\widetilde{f}}$ and whose quantum query complexity is $poly(n)$, whereas its classical query complexity is superpolynomial.*

Proof. The proof of the lower bound on the classical query complexity for $\mathcal{O}$ is information theoretic. The tightness of the bound follows since n bits of information about s have to be gathered and each query can yield at most 1 bit. To see that $O(n)$ are indeed sufficient, consider the following (adaptive) strategy for finding a shift (s, s') of $g(x, y) = (x+s)\pi(y+s')$: first query $g(x, y)$ on $(0, 0)$ to extract $s\pi(s')$. Then subtract this from the values at the points $(e_i, 0)$, where e_i denotes the ith standard basis vector. This gives the bits of $\pi(s')$. Next evaluate $\widetilde{f}(x, y) = \pi^{-1}(x)y^t$ at the points $(\pi(s'), e_i)$. This gives the bits of s'. Finally, from evaluating g at points $(0, \pi^{-1}(e_i) + s')$ we can obtain the bits of s, i. e., the entire hidden shift (s, s').

A standard argument can be invoked [BV97] to recursively construct an oracle which hides a function computed by a tree, the nodes of which are given by the oracle hiding a string s. In order to evaluate $f(x)$ at a node, first a sequence of smaller instances of the problem have to be solved. We do not go into further detail of the construction and only note that we get the analogous result as in [BV97], see also [HH08], namely that a tree of height $\log n$ leads to a quantum query complexity of $2^{\log n}$ which is polynomial in n, whereas the classical query complexity is given by $n^{\log n}$ which grows faster than any polynomial. $\square$

The following theorem avoids the adaptive queries in the proof of Theorem 8 and uses oracles of the form $\mathcal{O}_f$ in which no queries to the dual bent function are allowed. Since the quantum computer can still determine the shift in polynomial time, here an exponential separation between classical and quantum query complexity can be shown.

Theorem 9. *Let $\mathcal{O}_f$ be an oracle that hides a hidden shift s for an instance (f, g) of a hidden shift problem for a bent function f from Maiorana-McFarland class. Then classically $\Theta(\sqrt{2^n})$ queries are necessary and sufficient to identify the hidden shift s.*

Proof. The proof is similar to the lower bound for the linear structure problem considered in [dBCW02] and the query lower bound for Simon's problem [Sim94]. First, note that we can use Yao's minimax principle [Yao77] to show limitations of a deterministic algorithm $\mathcal{A}$ on the average over an adversarially chosen distribution of inputs. Hence, we can consider deterministic algorithms and π and s in the definition of $f(x, y) = x\pi(y)^t$ and $g(x, y) = f(x, y + s)$ will be chosen randomly.

The distribution we chose to show the lower is to chose π uniformly at random in S_{2^n}, the symmetric group on the strings of length n, and $s = (s_1, s_2) \in \mathbb{Z}_2^{2n}$ such that $s_1 = 0$ and s_2 is chosen uniform at random in $\mathbb{Z}_2^n$. The instances we consider are given by oracle access to the functions $f(x, y) = x\pi(y)^t$ and $g(x, y) = f(x, y + s) = x\pi(y + s)^t$. Now, without loss of generality we can assume that the classical algorithm $\mathcal{A}$ has (adaptively or not) queried the oracle $k = n^{O(1)}$ times, i. e., it has chosen pairs (x_i, y_i) for $i = 1, \ldots, k$ and obtained results

$$\begin{aligned} x_i\pi(y_i)^t &= a_i \\ x_i\pi(y_i + s)^t &= b_i. \end{aligned}$$

In order to characterize the information about s after these k queries we define set $D = \{x_i : i = 1, \ldots, k\} \cup \{y_i : i = 1, \ldots, k\}$. We show that if no collision between the values of f and g was produced, then the information obtained about s is exponentially small. To simplify our argument, we actually make the classical deterministic algorithm more powerful by giving oracle access to $\pi(x)$ and $\pi(x + s)$. Consider the set of all differences $D^{(-)} = \{d_1 - d_2 : d_1, d_2 \in D\}$ and the set $D_{good} = \mathbb{Z}_2^n \setminus D^{(-)}$. Note that for an abelian group A and subset $D \subset A$ with $|D|^2 < |A|$ we can always choose a set S such that $D \cap (D + s) = \emptyset$ for all $s \in S$. Indeed, we can choose $S = D_{good}$ since $x \in D \cap (D + s)$ would imply that there exist $d_1, d_2 \in D$ with $d_1 = d_2 + s$, i. e., $s \in D^{(-)}$ which is a contradiction. Notice in our case that $|S| \geq 2^n - |D^{(-)}| = 2^n - n^{O(1)}$. Now, we can change the value of the shift s to any other value s' as long as the algorithm has not queried s directly (the chances of which are exponentially small: because of a birthday for the strings s, the probability is given by $\Theta\left(\frac{1}{\sqrt{2^n}}\right)$). We do this by choosing π' in such a way that it maps $\pi(y_i + s) = \pi'(y_i + s')$ while being consistent with all other queries. Because of the above argument, as long as there is no collision, after ℓ queries to f, g, we still have a set S of size $|S| \geq 2^n - n^{O(1)}$ of candidates s', and π' which are also consistent with the sampled data, showing the lower bound. $\square$

Corollary 10. *There exists an oracle $\mathcal{O}$ implementing a Boolean function such that* $\mathsf{P}^{\mathcal{O}} \neq \mathsf{BQP}^{\mathcal{O}}$.

5 Conclusions

We introduced the hidden shift problem for a class of Boolean functions which are at maximum distance to all linear functions. For these so-called bent functions the hidden shift problem can be efficiently solved on a quantum computer, provided that we have oracle access to the shifted version of the function as well as its dual bent function. The quantum computer can extract the hidden shift using just one query to these two functions and besides this only requires to compute the Hadamard transform and measure qubits in the standard basis. We showed that this task is significantly more challenging for a classical computer and proved an exponential separation between quantum and classical query complexity.

Acknowledgments

The author gratefully acknowledges support by ARO/NSA under grant W911NF-09-1-0569 and would like thank the anonymous referees for valuable comments on this paper and earlier versions of it.

References

[Aar03] S. Aaronson. Quantum lower bound for recursive Fourier sampling. *Quantum Information and Computation*, 3(2):165–174, 2003.

[AS07] A. Atici and R. Servedio. Quantum algorithms for learning and testing juntas. *Quantum Information Processing*, 6(5):323–348, 2007.

[BCvD05] D. Bacon, A. Childs, and W. van Dam. From optimal measurement to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 469–478, 2005.

[dBCW02] N. de Beaudrap, R. Cleve, and J. Watrous. Sharp quantum versus classical query complexity separations. *Algorithmica*, 34(4):449–461, 2002.

[BJL99] Th. Beth, D. Jungnickel, and H. Lenz. *Design Theory*, volume I. Cambridge University Press, 2nd edition, 1999.

[BV97] E. Bernstein and U. Vazirani. Quantum complexity theory. *SIAM Journal on Computing*, 26(5):1411–1473, 1997. Conference version in Proc. STOC'93, pp. 11–20.

[CG06] C. Carlet and Ph. Gaborit. Hyper-bent functions and cyclic codes. *Journal of Combinatorial Theory, Ser. A*, 113:466–482, 2006.

[CvD07] A. Childs and W. van Dam. Quantum algorithm for a generalized hidden shift problem. In *Proceedings of the 18th Symposium on Discrete Algorithms (SODA'07)*, pages 1225–1232, 2007.

[CvD08] A. Childs and W. van Dam. Quantum algorithms for algebraic problems. arXiv Preprint 0812.0380, to appear in Reviews of Modern Physics.

[CSV07] A. Childs, L. J. Schulman, and U. Vazirani. Quantum algorithms for hidden nonlinear structures. In *Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07)*, pages 395–404, 2007.

[CW07] A. Childs and P. Wocjan. On the quantum hardness of solving isomorphism problems as nonabelian hidden shift problems. *Quantum Information and Computation*, 7(5–6):504–521, 2007.

[vDHI03] W. van Dam, S. Hallgren, and L. Ip. Quantum algorithms for some hidden shift problems. In *Proceedings of the 14th Symposium on Discrete Algorithms (SODA'03)*, pages 489–498, 2003.

[DDW08] Th. Decker, J. Draisma, and P. Wocjan. Efficient quantum algorithm for identifying hidden polynomials. *Quantum Information and Computation*, 2008. To appear, see also arxiv preprint 0706.1219".

[DJ92] D. Deutsch and R. Jozsa. Rapid solution of problems by quantum computation. *Proceedings of the Royal Society London, Series A*, 439:553–558, 1992.

[Dil75] J. Dillon. Elementary Hadamard difference sets. In F. (et al.) Hoffman, editor, *Proc. 6th S-E Conf. on Combinatorics, Graph Theory, and Computing*, pages 237–249. Winnipeg Utilitas Math., 1975.

[Dob95] H. Dobbertin. Construction of bent functions and balanced Boolean functions with high nonlinearity. In B. Preneel, editor, *Fast Software Encryption*, volume 1008 of *LNCS*, Springer, pages 61–74, 1995.

[FIM+03] K. Friedl, G. Ivanyos, F. Magniez, M. Santha, and P. Sen. Hidden translation and orbit coset in quantum computing. In *Proc. STOC'03*, pages 1–9, 2003.

[Hal02] S. Hallgren. Polynomial-time quantum algorithms for Pell's equation and the principal ideal problem. In *Proc. STOC'02*, pages 653–658, 2002.

[HH08] S. Hallgren and A. Harrow. Superpolynomial speedups based on almost any quantum circuit. In *Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP'08)*, pages 782–795, 2008.

[HMR+06] S. Hallgren, C. Moore, M. Rötteler, A. Russell, and P. Sen. Limitations of quantum coset states for graph isomorphism. In *Proceedings of the 38th Annual ACM Symposium on Theory of Computing (STOC'06)*, pages 604–617, 2006.

[HZ99] T. Helleseth and V. Zinoviev. On Z_4-linear Goethals codes and Kloosterman sums. *Designs, Codes and Cryptography*, 17:269–288, 1999.

[Kit97] A. Yu. Kitaev. Quantum computations: algorithms and error correction. *Russian Math. Surveys*, 52(6):1191–1249, 1997.

[KS07] A. R. Klivans and A. A. Sherstov. Unconditional lower bounds for learning intersections of halfspaces. *Machine Learning*, 69(2–3):97–114, 2007.

[Kup05] G. Kuperberg. A subexponential-time quantum algorithm for the dihedral hidden subgroup problem. *SIAM Journal on Computing*, 35(1):170–188, 2005.

[LW90] G. Lachaud and J. Wolfmann. The weights of the orthogonals of the extended quadratic binary Goppa codes. *IEEE Transactions on Information Theory*, 36(3):686–692, 1990.

[Man94] Y Mansour. Learning Boolean functions *via* the Fourier transform. In V. P. Roychodhury, K.-Y. Siu, and A. Orlitsky, eds., *Theoretical Advances in Neural Computation and Learning*, pp. 391–424. Kluwer, 1994.

[LMS08] S. Lovett, R. Meshulam, and A. Samorodnitsky. Inverse conjecture for the Gowers norm is false. In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC'08)*, pages 547–556, 2008.

[MS77] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error–Correcting Codes*. North–Holland, Amsterdam, 1977.

[Mon09] A. Montanaro. Quantum algorithms for shifted subset problems. *Quantum Information and Computation*, 9(5&6):500–512, 2009.

[MRRS07] C. Moore, D. N. Rockmore, A. Russell, and L. J. Schulman. The power of strong Fourier sampling: quantum algorithms for affine groups and hidden shifts. *SIAM Journal on Computing*, 37(3):938–958, 2007.

[NC00] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[Rot76] O. S. Rothaus. On "bent" functions. *Journal of Combinatorial Theory, Series A*, 20:300–305, 1976.

[RS04] A. Russell and I. Shparlinski. Classical and quantum function reconstruction via character evaluation. *Journal of Complexity*, 20(2–3):404–422, 2004.

[Sho97] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997.

[Sim94] D. R. Simon. On the power of quantum computation. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science (FOCS'94)*, pages 116–123, 1994.

[dW08] R. de Wolf. A brief introduction to Fourier analysis on the Boolean cube. *Theory of Computing Library–Graduate Surveys*, 1:1–20, 2008. Available online from www.theoryofcomputing.org.

[Yao77] A. Yao. Probabilistic computations: toward a unified measure of complexity. In *Proceedings of the 18th Annual Symposium on Foundations of Computer Science (FOCS'77)*, pages 222–227, 1977.